Assessing the Accuracy of Navigation Algorithms Using a Combined System of GPS Satellites and Pseudolites

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Abstract
The purpose of this work is to assess the accuracy of different navigation algorithms utilized for a combined system of GPS satellites and pseudolites. The algorithms under investigation are the ordinary pseudorange least square algorithm (LSA), single difference and double difference Kalman based algorithms, and the newly derived algorithm which employs the Doppler measurements. An integrity measure of the combined system for the different algorithms is also provided along with the navigation accuracy. A list of pros and cons for different algorithms follows the last section of the paper.

Introduction
The design and implementation of a robust, highly accurate filter has been the focus of our research in recent years. Initially, we attempted to design a filter, which would perform a CAT III precision landing utilizing only a system of pseudolites [1-2]. The filter design was based upon the technique of double difference and thus required a minimum of two receivers and two satellites in view in order to have one independent measurement all the time. The evaluation of the probability of false alarm and miss-detection was an important part of the receiver design. He was previously with the Networks and Comm. Division of Digital Corporation where he worked as a project engineer.
system design and its overall performance. Although the navigation performance was exceeding the FAA requirements for CAT III landing and the integrity performance was close to the corresponding FAA requirements, the complexity involved with processing data from two separated receivers is usually undesirable. In order to improve the system integrity and assure system operation even under severe geometry, weather, and interference environments we considered a combined system of satellites and pseudolites [3-5].

In our research on the indoor geo-location applications, we utilized a Kalman based filter to process indoor geo-location data [6-7]. This work constitutes our base line or serves as the corner stone. In an attempt to reduce the system complexity, introduce something novel, and at the same time maintain the system accuracy, we resurrected the Doppler based navigation idea [8], which is being presented in [9-10].

This paper, thus serves as the synthesis of our research to obtain an easily implemental, highly accurate filter when a single receiver processes all measurements on the fly. The following section discusses all the navigation algorithms researched and implemented. Next, we proceed with the integrity measurement and the simulation section. The conclusions and methods' assessment section follows after the simulation. Finally, a list of useful references is provided at the end of the paper.

**Navigation Algorithms**

Although most of the navigation algorithms are discussed previously [1-10], the intent here is to consider their design from the point of view of a system with combined satellites and pseudolites. We have divided navigation algorithms into three categories: LSA pseudorange based, accumulated carrier Kalman based, and newly derived Doppler LSA based.

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**Ordinary Pseudorange Least Square**

Consider a system of combined pseudolites and satellites as shown in figure 1. For the reasons explained in [9-10], we will no longer deal with the snap shot solution (except when the distances between the user and the source are within one sigma value of the measurement noise); therefore, consider the incremental solution formulation for the $k^{th}$ epoch and for either the satellites (S) or pseudolites (P) as,

$$
\Delta s_i^k = \left( H_i^{kT} R_i^{k-1} H_i^k \right)^{-1} H_i^{kT} R_i^{k-1} r_i^k , \ i = \{S, P\}. \quad (1)
$$

where,

- $\Delta s_i^k$ is the increment in the state vector,
- $H_i^k$ is the matrix which relates the residual vector with the state vector,

$$
R_i^k \quad \text{is the pseudorange measurement noise covariance matrix},
$$

$$
r_i^k \quad \text{is the residual vector determined from},
$$

$$
r_i^k = \rho_i^k - \hat{d}_i^k \left( \hat{s}_i^{k-1} \right) , \ i = \{S, P\}. \quad (2)
$$

In expression (2), $\rho_i^k$ denotes the raw pseudorange measurement vector and $\hat{d}_i^k$ is the range estimate as a function of the state vector, $\hat{s}_i^k$. 

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*Figure 1: Combined SAT/PSL system without reference station*
We have assumed that the receiver is capable of tracking both pseudolite and satellite signals.

If we assume that in the beginning the filter processes satellite data then the solution for the state vector is of the form,

\[
\hat{s}_S^1 = s_0^0 + \Delta\hat{s}_S^1, \quad (3)
\]

where \( s_0^0 \) is the starting state vector. Next, the filter will process pseudolite data; therefore, the state solution is given by,

\[
\hat{s}_P^1 = \hat{s}_S^1 + \Delta\hat{s}_P^1. \quad (4)
\]

The solution (4) will serve as the starting point for the satellite solution. Based on this argument, the solution for the state vector at the \( k^{th} \) epoch can be written as,

\[
\hat{s}_S^k = \hat{s}_P^{k-1} + \Delta\hat{s}_S^k, \quad (5)
\]

and state solution vector for processing pseudolite data,

\[
\hat{s}_P^k = \hat{s}_S^k + \Delta\hat{s}_P^k. \quad (6)
\]

The following diagram illustrates the sequence of the measurement processing and the state vector updates. One can easily reconstruct the combined PSL/SAT filter, if the filter state is updated first based on pseudolite data.
Single Difference and Double Difference

Point positioning and kinematic positioning can be accomplished using carrier phase and the technique of single or double differencing. Whichever the technique applied, the satellite with the highest elevation and the pseudolite with the lowest elevation are selected as the base satellite and pseudolite respectively. A detailed description of the system is given in figure 3.

A Kalman based filter utilizing single difference accumulated phase measurement was derived in [6]. The same filter will be generalized for a system of combined SAT/PSL data. Thus, based on equation (7) of [6], the single difference accumulated carrier phase vector can be written as,

\[ \overline{\varphi}_i^k = \overline{d}_i^k + \lambda \overline{a}_i^0 + 2 \overline{e}_i^k, \quad i = \{S, P\}. \]  (7)

Here, \( \overline{\varphi}_i^k \) is the single difference accumulated phase vector, \( \overline{d}_i^k \) is the estimated single difference range, \( \lambda \overline{a}_i^0 \) denotes the modified single difference ambiguities (see (4) of [6]), \( \overline{e}_i^k \) is the error term, and \( \lambda \) of course denotes the wavelength of the carrier.

Similar to the single difference equation, one can formulate the double difference accumulated carrier phase measurement vector, \( \overline{\varphi}_i^k \), as [7],

\[ \overline{\varphi}_i^k = \overline{d}_i^k + \lambda \overline{a}_i^0 + 2 \overline{e}_i^k, \quad i = \{S, P\}. \]  (8)

Under the assumption that \( \varphi_i^k \) can be either the single difference, \( \overline{\varphi}_i^k \), or the double accumulated phase, \( \overline{\varphi}_i^k \), the diagram of the Kalman based filter is pictured in figure 4. More details of the double difference technique can be found in [11].

![Figure 4: Combined SAT/PSL single/double difference filter](image)

Newly Doppler Least Squares

Assuming an initial condition for the state vector \( s_0^0 \), the Doppler derived pseudorange can be formed as [9-10],

\[ \hat{\rho}_i^k = \hat{d}_i^{k-1} + \varphi_i^{k-1} + 0.5(\varphi_i^{k-1} - \varphi_i^{k-2}) + w_i^{k-1}. \]  (9)
In the above expression, the explanation of the unknowns reads:

\( \hat{\rho}_k \) is the Doppler derived pseudorange;

\( \hat{d}_k \) is the estimate user-source range;

\( \hat{\phi}_k \) is the Doppler between the user and source;

\( \hat{w}_k \) denotes the process noise;

subscript \( i \) is the index for satellite (S) or pseudolite (P) respectively;

superscript \( k \) is the epoch index.

Although in [9] we did not include the second derivative of the user-source range, we have included that here assuming that the trajectory includes acceleration or higher order derivatives.

At this point we should be able to modify the filter diagram 1 to reflect relation (7). While constructing the filter, which processes the SAT and PSL data, we have assumed that the filter processes first the SAT data and then the PSL data.

**Integrity**

In order to provide a means for integrity we will consider the residual formulation,

\[
    r_i^k = m_i^k - \hat{m}_i^k(s_i^{k-1}), \quad i = \{S, P\}, \quad (10)
\]

where, \( m_i^k \) is the measurement vector and \( \hat{m}_i^k(s_i^{k-1}) \) is the nonlinear estimate of the measurement vector \( m_i^k \).

Once the incremental solution update is obtained from,

\[
    \hat{s}_i^k = \hat{s}_i^{k-1} + \Delta \hat{s}_i^k, \quad (11)
\]

\[
    \Delta \hat{s}_i^k = \left( H_i^k R_i^{-1} H_i^k \right)^{-1} H_i^k R_i^{-1} r_i^k, \quad (12)
\]
The residual vector is recomputed from,
\[ r_i^{k+} = m_i^k - \hat{m}_i^k (s_i^k), \]
\[ = m_i^k - \hat{m}_i^k (s_i^k + \Delta e_{si}^k), \]
\[ = r_i^k + \Delta e_{ri}^k. \] (13)

The nonlinear relation between the error in the residual vector, \( \Delta e_{ri}^k \), and the solution increment, \( \Delta e_{si}^k \), is given by the following set of measures,

- **Probability of False Alarm**
  It is the probability that the normalized second norm of the residual error is greater than the residual threshold under normal conditions [12],
  \[ P_{FA} = P\left( \frac{\|\Delta e_{ri}^k\|}{N_r} > \tau_r \mid NC \right), \] (14)
  where \( N_r \) is the size of the residual vector, \( \tau_r \) is the threshold of the residual vector, and NC denotes normal condition.

- **Probability of Misdetection**
  It is the probability that the normalized norm of the residual threshold is smaller than the residual threshold at the same time the solution error is greater than the accepted solution accuracy [12], as
  \[ P_{MD} = P\left( \frac{\|\Delta e_{ri}^k\|}{N_r} < \tau_r \cap \frac{\|\Delta e_{si}^k\|}{N_s} < \tau_s \right), \] (15)
  where \( N_s \) is the size of the sub state vector. \( \tau_s \) is the solution accuracy threshold.

**Simulation**

The simulation scenario is the same with the one corresponding to the moving firefighter in [9-10].

**Modified Pseudorange LSA**

In figures 6 and 7 we present the lateral and vertical position error (LPE and VPE) vs. pseudorange measurement noise for the modified LSA algorithm.
Single-Difference Kalman Based

Kalman filter can be driven using either one of the measurements (pseudorange, accumulated carrier phase, or Doppler) or their combination.

When Kalman filter is driven with pseudorange only measurements, the lateral and vertical position and velocity errors (LPE, VPE, LVE, and VVE) vs. distance are pictured in figures 8 through 11.
When Kalman filter is driven with pseudorange and carrier phase measurements, the lateral and vertical position and velocity errors vs. distance are pictured in figures 12 through 15.

Newly Derived Doppler

Similar to the pseudorange only case, we process Doppler measurements using the modified LSA.

The lateral and vertical position errors vs. Doppler measurement error are shown in figures 16 and 17.

Conclusions and Methods’ Assessment

For the system of combined pseudolites and satellites, we present the filter design assessment according to the accuracy (or navigation error mean and standard deviation in parentheses) in Table 1.

According to the results presented in Table 1, it appears that the Kalman engine performs better than either one of the MLSAs.

If we split the cost of the filter in the cost of designing (CD) and the cost of tuning (CT), then the Kalman filter cost is higher than the MLSA.

Table 1: Methods’ assessment according to accuracy

<table>
<thead>
<tr>
<th>Method</th>
<th>MLSA (PR)</th>
<th>Kalman (PR)</th>
<th>Kalman (PR&amp;CP)</th>
<th>MLSA (DO)</th>
</tr>
</thead>
<tbody>
<tr>
<td>LPE (m)</td>
<td>12 (15)</td>
<td>4 (4)</td>
<td>0.1 (0.4)</td>
<td>1.2 (0.5)</td>
</tr>
<tr>
<td>VPE (m)</td>
<td>0 (5)</td>
<td>1 (1.5)</td>
<td>0.1 (0.25)</td>
<td>0.25 (0.1)</td>
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<tr>
<td>LVE (m/s)</td>
<td>N/A</td>
<td>0.1 (0.3)</td>
<td>0.0 (0.00)</td>
<td>N/A</td>
</tr>
<tr>
<td>VVE (m/s)</td>
<td>N/A</td>
<td>0.01 (0.3)</td>
<td>0.0 (0.00)</td>
<td>N/A</td>
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Table 2: Methods’ accuracy according to cost

<table>
<thead>
<tr>
<th>Method</th>
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<th>Kalman (PR)</th>
<th>Kalman (PR&amp;CP)</th>
<th>MLSA (DO)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost</td>
<td>CD</td>
<td>CD+CT</td>
<td>CD+CT</td>
<td>CD</td>
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References


