ADJUSTMENT OF OBSERVATIONS BY LEAST SQUARES

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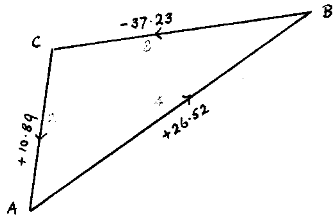
1. Introduction

In any practical problem, if more observations have been taken than are necessary for the calculation of the required quantities, then from these observations different sets of the minimum number needed could be selected and, for each set, different values of the final parameters would be obtained. A method has to be devised of making adjustments to all observations in order to obtain the best possible estimates of the quantities we are trying to find.



To take a very simple example, consider a level traverse run from a point A of known height to fix the heights of two other points B & C and let the height difference observations be as shown in the diagram.

Only two out of these three observed differences in height are necessary for the fixation of the heights of B & C and the inclusion of the third observation results in two alternative heights for the two points. Let v_1, v_2 & v_3 be the corrections to be made to the difference in heights for the lines AB, BC & CA respectively then since on returning to A we must have zero difference in height

$$(26.52 + v_1) + (-37.23 + v_2) + (10.89 + v_3) = 0$$

or $$v_1 + v_2 + v_3 + .18 = 0$$

This single equation in three unknowns has an infinite number of solutions and our choice of the method of solution can be quite arbitrary. We could make equal corrections of -.06 to each of the observed differences in height, or we could assume that the error in each height difference was proportional to the length of the line

traversed (4, 3 and 2 km respectively) which would give corrections of -.08, -.06 and -.04. It should be stressed here that whatever corrections we make, the probability that those corrections cancel out the true errors is very small, even for a simple figure of this nature and with more complicated networks the probability will be minute.

In applying the Principle of Least Squares, we do not aspire to obtain the correct values, which are unfortunately always unknown, of the required quantities but only to obtain answers which it is hoped will be, on average, closer to the true values than those obtained by other methods. It also has the advantage that a unique answer is obtained from the calculations, unlike the semi-graphic method where the finally accepted result is dependent on the judgement of the computer.

2. The Principle of Least Squares is based on the assumption that if a number of measurements of a quantity are taken, the most probable value of that quantity is the mean of the individual measurements. Suppose n individual measurements $x_1, x_2, x_3 \ldots\ldots x_n$ of a quantity have been made then if the mean value is denoted by $\bar{x}$

$$n\bar{x} = \left[x_i\right]_1^n \qquad 2.1$$

Let v_i be the difference between any particular x_i and the mean $\bar{x}$

$$\left[v_i^2\right]_1^n = \left[(x_i - \bar{x})^2\right]_1^n = \left[x_i^2 - 2\bar{x}x_i + \bar{x}^2\right]_1^n$$

$$= \left[x_i^2\right]_1^n - 2\bar{x}\left[x_i\right]_1^n + n\bar{x}^2$$

$$= \left[x_i^2\right]_1^n - n\bar{x}^2 \qquad 2.2$$

Now let w_i be the difference between x_i and some number $X \neq \bar{x}$

$$\left[w_i^2\right]_1^n = \left[(x_i - X)^2\right]_1^n = \left[x_i^2\right]_1^n - 2X\left[x_i\right]_1^n + nX^2$$

$$= \left[v_i^2\right]_1^n + n\bar{x}^2 - 2nX\bar{x} + nX^2 \qquad \text{from 2.1 \& 2.2}$$

$$= \left[v_i^2\right]_1^n + n(\bar{x} - X)^2 \qquad 2.3$$

As $(\bar{x} - X)^2$ is always positive $\left[w_i^2\right]_1^n > \left[v_i^2\right]_1^n$ i.e. the sum of the squares of the residuals from the mean is less than the sum of the squares of the residuals from any other value.

Conversely if we are told that $\left[w_i^2\right]_1^n$, the sum of the squares of the residuals from some value X is a minimum for any set of observations, then by equation 2.3 we can say that $(\bar{x} - X) = 0$ i.e. X is the mean value. Hence assuming that the mean is the best value and applying the principle of Least Squares constitute one and the same operation. It will later be demonstrated that all the Method of Least Squares does is to obtain, by a simpler method, the means of the corrections which would have been applied to the individual observations, if all possible sets of minimum data were separately considered.

If in the series of measurements, the measurement x_i does not occur once only but occurs n_i times then the formulae given above become

$$\bar{x}\left[n_i\right] = \left[n_i x_i\right] \qquad 2.1a$$

and

$$\left[n_i v_i^2\right] = \left[n_i x_i^2\right] - \bar{x}^2\left[n_i\right] \qquad 2.2a$$

and the number n_i can be called the weight of the measurement x_i

In the above it has been assumed that the weight is a whole number but this is not a necessary condition as if all the weights were multiplied by some arbitrary constant C equation 2.1a would become

$$\bar{x} = \frac{\left[Cn_i x_i\right]}{\left[Cn_i\right]} = \frac{\left[n_i x_i\right]}{\left[n_i\right]} \quad \text{as before}$$

and hence, for the purpose of obtaining the mean, it is the relative values of the weights and not their absolute values that matter.

If n measurements $x_1, x_2, x_3 \ldots . x_n$ of a quantity have been made the standard deviation of a single observation is given by

$$\sigma^2 = \frac{\left[v_i^2\right]_1^n}{n} \quad \text{or} \quad \frac{\left[n_i v_i^2\right]_1^n}{\left[n_i\right]_1^n} \qquad 2.4$$

and if σ_o is the standard deviation of the mean of these observations

$$\sigma_o^2 = \frac{\sigma^2}{n} \quad \text{or} \quad \frac{\sigma^2}{\left[n_i\right]_1^n} \qquad 2.5$$

Both the weights and the standard deviations are measures of the reliability of the measurements, good observations corresponding to high weights or small standard deviations and v.v. It is thus evident that the weight must be inversely proportional to the standard deviation or to some power of it. If an observation, of standard deviation σ_1 has a weight n_1, where n_1 is a whole number, we can regard this observation as the mean of n_1 standard observations (of some standard deviation σ) and hence $\sigma^2 = n_1\sigma_1^2$

Similarly for another observation of weight n_2 and standard deviation σ_2 $\qquad \sigma^2 = n_2\sigma_2^2$

$$\therefore \quad \frac{n_1}{n_2} = \frac{\sigma_2^2}{\sigma_1^2}$$

or the weights of the two observations are inversely proportional to the squares of their standard deviations.

Thus from 2.2a, the function which we want to minimise is

$$\left[n_i v_i^2\right] \quad \text{or} \quad \left[\frac{v_i^2}{\sigma_i^2}\right]$$

σ_i^2, the square of the standard deviation, is called the variance.

3. <u>Extension to more than one variable</u>.

This is most easily explained by consideration of the Normal Curve of Error

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-v^2}{2\sigma^2}} \qquad 3.1$$

where y is the probability of the occurence of an error of magnitude lying between v and (v + dv) from the mean.

If measurements have been taken of n independent variables, the probability of a given set of errors occurring simultaneously will be

$\prod_{i=1}^{n} p_i$ where p_i are the individual probabilities.

$$\therefore \text{ probability of the set occurring} = \frac{1}{(2\pi)^{\frac{n}{2}} \prod_{i=1}^{n}\sigma_i} e^{-\left[\frac{v_i^2}{2\sigma_i^2}\right]_1^n}$$

To obtain the set with the highest probability of occurring, this expression must be made a maximum and since the only factors that we can change are the v's, this means that $\left[\dfrac{v_i^2}{2\sigma_i^2}\right]_1^n$ must be made a minimum, the same condition as was obtained for a single variable.

4. Methods of carrying out a Least Square Adjustment

For a least square adjustment to be carried out, it must be possible to derive a set of linear equations connecting the parameters, which it is required to find, with the observed quantities (after adjustment). In some cases this is a straightforward process as, for instance, when a level net has been observed, but planimetric co-ordinates are not linear functions of the observations of directions and distances from which they are obtained and in this case the relationship has to be linearised by calculating approximate values of the parameters and using a Taylor's Series expansion as far as the first order terms.

The general form of the equations contains more than one parameter and more than one observation but this form can always be reduced to two simpler forms which are more easily dealt with. They are (a) Observation or Parametric Equations which are equations connecting one or more unknowns with a single observation. There will thus be one equation for every direction, distance etc. observed. If it is required to fix n variables then we must have a minimum of n observations and for redundant observations there will be m observations, and hence m equations in n unknowns where $m > n$ (b) Condition Equations which express the geometric conditions which must be satisfied by the adjusted values of the observations. If, as before, m observations have been taken to fix n variables, then there will be (m-n) conditions which the adjusted values of the observations must satisfy. In this case there will be fewer equations than observations, i.e. (m-n) equations in m corrections to the observations and hence the two types of equation will need different methods of treatment.

The two methods are connected however and give the same final results and it will be shown later that the condition equations are the equations remaining after the parameters have all been eliminated from the parametric equations.

In the following sections the theory will be dealt with in both the classical and Matrix notations, as the subsequent error analysis is very easily dealt with by Matrix methods but would be very cumbersome to derive by classical algebra. Appendix A gives a short resume of all the Matrix Algebra theory which is needed to follow the derivations and Appendix B gives details of the methods generally adopted for the solution of the sets of linear equations.

5. Observation or Parametric Equation Method.

Suppose that a series of observations $p^T = \left[p_1 p_2 p_3 \;\ldots . p_m \right]$ have been taken to fix a set of variables $X^T = \left[x_1 x_2 x_3 \;\ldots . x_n \right]$ where $m > n$, then if $V^T = \left[v_1 v_2 v_3 \;\ldots . v_m \right]$ are the corrections required to give estimates of the true (but unknown) values $P^T = \left[P_1 P_2 P_3 \;\ldots . P_m \right]$ which should have been obtained for the observations.

$$P_i = p_i + v_i \quad i = 1 \text{ to } m \qquad\qquad P = p + V$$

and we shall have m equations of the form

$$\begin{array}{ll}
a_{11}x_1 + a_{12}x_2 + \ldots . \quad + a_{1n}x_n + K_1 - P_1 = 0 & \\
a_{21}x_1 + a_{22}x_2 + \ldots . \quad + a_{2n}x_n + K_2 - P_2 = 0 & AX + K - P = 0 \qquad 5.1 \\
a_{31}x_1 + a_{32}x_2 + \ldots . \quad + a_{3n}x_n + K_3 - P_3 = 0 & \\
\;\cdot\;\cdot\;\cdot\;\cdot \qquad\qquad \cdot\;\cdot\;\cdot\;\cdot & \\
\;\cdot\;\cdot\;\cdot\;\cdot \qquad\qquad \cdot\;\cdot\;\cdot\;\cdot & \\
a_{m1}x_1 + a_{m2}x_2 + \ldots . \quad + a_{mn}x_n + K_m - P_m = 0 &
\end{array}$$

or if we put $c_i = K_i - p_i \quad i = 1$ to m $\qquad\qquad C = K - p$

$$\begin{array}{ll}
a_{11}x_1 + a_{12}x_2 + \ldots . \quad + a_{1n}x_n + c_1 = v_1 & \\
a_{21}x_1 + a_{22}x_2 + \ldots . \quad + a_{2n}x_n + c_2 = v_2 & AX + C = V \qquad 5.2 \\
\;\cdot\;\cdot\;\cdot\;\cdot \qquad\qquad \cdot\;\cdot\;\cdot\;\cdot\;\cdot & \\
\;\cdot\;\cdot\;\cdot\;\cdot \qquad\qquad \cdot\;\cdot\;\cdot\;\cdot\;\cdot & \\
a_{m1}x_1 + a_{m2}x_2 + \ldots . \quad + a_{mn}x_n + c_m = v_m &
\end{array}$$

where A is an (m,n) coefficient matrix

X is an (n,1) vector of parameters

p is an (m,1) vector of unadjusted observations

V is an (m,1) vector of corrections

P is an (m,1) vector of adjusted observations

K & C are (m,1) vectors of constants

In general the observations will be of different standards of accuracy and to allow for this the i'th equation will be given a weight $1/g_{ii}$ where g_{ii} is proportional to σ_i^2. The Least Square condition that

$$F = \left[\frac{v_i^2}{g_{ii}} \right]_1^m = V^T G^{-1} V$$

is a minimum must now be applied, G being an (m,m) diagonal matrix, the diagonal elements of which are g_{ii}

$$F = \left[\frac{(a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j \ldots + a_{in}x_n + c_i)^2}{g_{ii}} \right]_1^m = (AX + C)^T G^{-1} (AX + C)$$

The conditions for F to be a minimum are $\dfrac{\partial F}{\partial x_j} = 0$

which gives n equations for j = 1 to j = n

$$\left[\frac{a_{ij}}{g_{ii}} (a_{i1}x_1 + a_{i2}x_2 + \ldots \quad + a_{ik}x_k + \ldots + a_{in}x_n + c_i) \right]_1^m = 0$$

Rearranging the terms this equation becomes

$$x_1 \left[\frac{a_{i1}a_{ij}}{g_{ii}} \right]_1^m + x_2 \left[\frac{a_{i2}a_{ij}}{g_{ii}} \right]_1^m + \ldots\ldots + x_n \left[\frac{a_{in}a_{ij}}{g_{ii}} \right]_1^m + \left[\frac{a_{ij}c_i}{g_{ii}} \right]_1^m = 0 \qquad 5.3$$

This is a series of n equations, in n unknowns, which can be solved to give a unique set of values for the parameters $x_1\ x_2\ \ldots\ x_n$.

These equations are called the Normal Equations and if they were all written out for j = 1 to j = n, it would be seen that the coefficient matrix of the equations is symmetrical. This fact enables us to adopt special methods for the solution of the equations.

Using Matrix Notation

$$F = (AX + C)^T G^{-1} (AX + C) = (X^T A^T + C^T) G^{-1} (AX + C)$$

$$= X^T A^T G^{-1} AX + X^T A^T G^{-1} C + C^T G^{-1} AX + C^T G^{-1} C$$

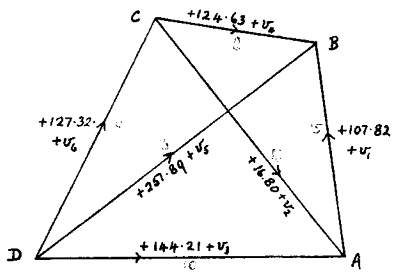
Since G is a diagonal matrix, G^{-1} and $(A^T G^{-1} A)$ will both be symmetrical and hence by Appendix A7

$$\frac{dF}{dX} = 2A^TG^{-1}AX + A^TG^{-1}C + A^TG^{-1}C = 2A^TG^{-1}(AX + C)$$

giving the Normal Equations $A^TG^{-1}AX + A^TG^{-1}C = O$ 5.3a

and the solution $X = -\ (A^TG^{-1}A)^{-1}A^TG^{-1}C$ 5.4

<u>Numerical Example</u>



Point A is a point of known height (1125.92) and the results of a levelling net to fix the heights of three other points B, C & D and the distances of the lines levelled are shown on the diagram. In levelling adjustments it is often assumed that the accuracy of a given difference in height is inversely proportional to the distance levelled and this weighting system will be used in this example, the corrections $v_1\ v_2 \ldots v_6$ to the observed differences in height being as shown on the diagram.

If the heights of the unknown points are H_B, H_C and H_D, then for the observation of the lines AB and BC the observation equations and their weights would be

$1125.92 + (107.82 + v_1) = H_B$ weight 1/15

$H_C + (124.63 + v_4) = H_B$ " 1/8

It will be noted that the relationship between the observations (the differences in height) and the variables (the heights) is linear and hence there is no necessity to obtain approximate values of the variables. In practice however it will be convenient to do so as this will reduce the size of the numbers involved in the calculations and some of the calculations will be simplified owing to some of the constant terms of the equations being zero.

Let $H_B = 1125.92 + 107.82 + x_1 = 1233.74 + x_1$

$H_C = 1125.92 - 16.80 + x_2 = 1109.12 + x_2$

$H_D = 1125.92 - 144.21 + x_3 = 981.71 + x_3$

The unknown variables are now x_1 x_2 x_3 instead of H_B, H_C & H_D and the observation equations become

Line	Equation	Weight
AB	$1125.92 + (107.82 + v_1) = 1233.74 + x_1$	1/15
AC	$1125.92 - (16.80 + v_2) = 1109.12 + x_2$	1/12
AD	$1125.92 - (144.21 + v_3) = 981.71 + x_3$	1/10
BC	$1233.74 + x_1 - (124.63 + v_4) = 1109.12 + x_2$	1/8
BD	$1233.74 + x_1 - (251.89 + v_5) = 981.71 + x_3$	1/16
CD	$1109.12 + x_2 - (127.32 + v_6) = 981.71 + x_3$	1/16

These equations are usually shown in tabular form as V = AX + C but without the v's being actually specified. Since when error analysis is not being carried out only the relative values of the weights are important, they will all be multiplied by 12 to make them on average close to unity. The observation equations would then be tabulated as follows.

Weight	x_1	x_2	x_3	C	S
.8	1	0	0	0	1.0
1.0	0	-1	0	0	-1.0
1.2	0	0	-1	0	-1.0
1.5	1	-1	0	-.01	-0.01
.75	1	0	-1	+.14	+0.14
.75	0	1	-1	+.09	+0.09

The zero co-efficients for the variables x_1 x_2 & x_3 can be omitted, if so desired, but not the zeros in the constant term column C. The S column, which is added to provide a check on the subsequent computations, is formed by adding all the figures, excluding the weight, in the same row.

Carrying out the process detailed in equation 5.3 gives the normal equations.

x_1	x_2	x_3	C	S
+3.05	-1.50	- .75	+.0900	+ .8900
-1.50	+3.25	- .75	+.0825	+1.0825
- .75	- .75	+2.70	-.1725	+1.0275

The way the figures in the first row are obtained is as follows.

+ 3.05 = .8x1x1 + 1.0x0 + 1.2x0 + 1.5x1x1 + .75x1x1 + .75x0

-1.50 = .8x1x0 + 1.0x0x(-1) + 1.2x0x0 + 1.5x1x(-1) + .75x1x0 + .75x0x1

- .75 = .8x1x0 + 1.0x0x0 + 1.2x0x(-1)+1.5x1x0 + .75x1x(-1)+.75x0x(-1)

+.0900 = .8x1x0 + 1.0x0x0+ 1.2x0x0 + 1.5x1x(-01)+.75x1x.14 + .75x0x.09

+.8900 = .8x1x1.0 + 1.0x0x(-1.0) +1.2x0x(-1.0)+ 1.5x1x(-.01)+ .75x1x.14+.75x0x.09

and all these calculations are checked by the fact that

+3.05 - 1.50 - .75 + .0900 = +.8900

Each row should be checked and if necessary recomputed before going on to the next one.

The equations tabulated above are those used in the example of a solution by the Cholesky method (Appendix B2) and on reference to this it will be found that the solutions are

$x_1 = -.0327$ $x_2 = -.0298$ $x_3 = +.0465$

$\therefore$ the required heights are B 1233.71, C 1109.06 and D 981.76

6. The Condition Equation Method or Method of Correlatives

The equations in this case express the geometric conditions that the adjusted observations P_i, and hence the corrections v_i, must satisfy. If as before m observations have been taken to fix n variables (co-ordinates, heights etc.) then there will be k = (m-n) independent conditions to be satisfied. After combination of constants these conditions will be of the form

$$\begin{aligned} b_{11}v_1 + b_{12}v_2 + \cdots\cdots &+ b_{1m}v_m + d_1 = 0 \\ b_{21}v_1 + b_{22}v_2 + \cdots\cdots &+ b_{2m}v_m + d_2 = 0 \qquad BV + D = 0 \qquad 6.1 \\ \vdots \qquad\qquad & \qquad \vdots \\ b_{k1}v_1 + b_{k2}v_2 + \cdots\cdots &+ b_{km}v_m + d_m = 0 \end{aligned}$$

and for the Least Square adjustment there is the additional condition that

$$F = \frac{v_1^2}{g_{11}} + \frac{v_2^2}{g_{22}} + \cdots\cdots + \frac{v_m^2}{g_{mm}} = V^TG^{-1}V \text{ is a minimum} \qquad 6.1a$$

In the above B is a (k,m) matrix of coefficients

D is a (k,1) vector of constants

and the other matrices and vectors are the same as those in Section 5.

Differentiating equations 6.1 and 6.1a gives

$$
\begin{array}{lll}
b_{11}dv_1 + b_{12}dv_2 + \ldots\ldots + b_{1m}dv_m = 0 & & \\
b_{21}dv_1 + b_{22}dv_2 + \ldots\ldots + b_{2m}dv_m = 0 & B(dV) = 0 & 6.2 \\
\cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \cdot & & \\
\cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \cdot & & \\
b_{k1}dv_1 + b_{k2}dv_2 + \ldots\ldots + b_{km}dv_m = 0 & & \\
\dfrac{v_1 dv_1}{g_{11}} + \dfrac{v_2 dv_2}{g_{22}} + \ldots\ldots + \dfrac{v_m dv_m}{g_{mm}} = 0 & (dV)^T G^{-1} V = 0 & 6.2a
\end{array}
$$

Now if we put $\dfrac{v_i}{g_{ii}} = b_{1i}\ell_1 + b_{2i}\ell_2 + \ldots\ldots + b_{ki}\ell_k \quad G^{-1}V = B^T L$ 6.3

where $\ell_1\ \ell_2\ \ell_3 \ldots \ell_k$ are constants (at present unknown), the satisfaction of equations 6.1 and 6.2 will automatically satisfy equation 6.2a and hence condition 6.1a

Substituting 6.3 in 6.1 gives the normal equations

$$
\begin{array}{l}
\ell_1\left[g_{ii}b_{1i}^2\right]_1^m + \ell_2\left[g_{ii}b_{1i}b_{2i}\right]_1^m + \ldots\ldots + \ell_k\left[g_{ii}b_{1i}b_{ki}\right]_1^m + d_1 = 0 \\
\ell_1\left[g_{ii}b_{1i}b_{2i}\right]_1^m + \ell_2\left[g_{ii}b_{2i}^2\right]_1^m + \ldots\ldots + \ell_k\left[g_{ii}b_{2i}b_{ki}\right]_1^m + d_2 = 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad BGB^T L + D = 0 \qquad 6.4 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \cdot \\
\ell_1\left[g_{ii}b_{1i}b_{ki}\right]_1^m + \ell_2\left[g_{ii}b_{2i}b_{ki}\right]_1^m + \ldots\ldots + \ell_k\left[g_{ii}b_{ki}^2\right]_1^m + d_k = 0
\end{array}
$$

This is a system of k equations in k unknowns, $\ell_1\ \ell_2 \ldots \ell_k$, which can be solved to give $L = -(BGB^T)^{-1}D$ 6.5

and on substitution of these values of L in equations 6.3, the required corrections are obtained.

$$V = GB^T L = -\ GB^T(BGB^T)^{-1}D \qquad 6.6$$

The constants $L^T = \ell_1\ \ell_2\ \ell_3 \ldots\ldots \ell_k$ are known as the Lagrangian Multipliers or Correlatives and the equations 6.3 are called the Correlate Equations.

These results can be derived by an alternative method which is given below in Matrix Notation only. Since the equations

6.1 are all equal to zero any multiple of them can be added to or subtracted from F without altering its value. Hence instead of minimising F, the same result can be obtained by minimising

$$F' = V^TG^{-1}V - 2L^T(BV + D)$$

where L is a (k,1) vector of constants

$\frac{\partial F'}{\partial V} = 2G^{-1}V - 2B^TL = 0$ giving equation 6.3 and hence the subsequent results.

Numerical Example

The same numerical example will be used as in Section 5. Since 6 observations have been taken to fix 3 new heights there will be (6 - 3) = 3 independent conditions. Going round any closed circuit, the total difference in height must be zero and of the possible circuits the most obvious ones and the conditions which arise from them are

Circuit	Condition
ABCDA	$v_1 + v_3 - v_4 - v_6 + .08 = 0$
ABCA	$v_1 + v_2 - v_4 - .01 = 0$
ACDA	$- v_2 + v_3 - v_6 + .09 = 0$
ABDA	$v_1 + v_3 - v_5 + .14 = 0$
BCDB	$- v_4 + v_5 - v_6 - .06 = 0$

Of these five conditions it will be seen that only 3 are independent as the first condition is equivalent to the sum of the second and third conditions and also to the sum of the fourth and fifth conditions. Any three may be selected for our condition equations provided they are independent e.g. the first three cannot be selected.

We will select the second, third and fourth conditions, changing the sign of the last condition in order to cut down some of the subsequent work. The equations will again be shown in tabular form and it should be noted that this tabulation serves two purposes as the rows give all the details of the condition equations whilst the columns give the details of the correlate equations.

	v_1	v_2	v_3	v_4	v_5	v_6	d
Weight	1.5	1.2	1.0	.8	1.6	1.6	
ℓ_1	+1	+1	0	-1	0	0	-.01
ℓ_2	0	-1	+1	0	0	-1	+.09
ℓ_3	-1	0	-1	0	+1	0	-.14
S	0	0	0	-1	+1	-1	

In this tabulation the first row under the headings gives the weights which are now proportional to g_{ii} and not to $1/g_{ii}$ as in the parametric method and in this case g_{ii} has been put equal to $(d_i/10)$. It should be stressed that this is not an alteration of the weighting system for the original observations; it merely adapts the same weighting system to give effect to a different method of calculation. The final row S is again a checking row for use in the subsequent calculations and it is formed by the sum of the other figures (excluding the weights) in the same column. It was in order to make as many of the elements of S zero as possible that the sign of the fourth condition was changed.

Applying the process of equations 6.4 gives the Normal.

Equations

ℓ_1	ℓ_2	ℓ_3	d	$\sum$
+ 3.5	- 1.2	- 1.5	- .01	+ .79
- 1.2	+ 3.8	- 1.0	+ .09	+ 1.69
- 1.5	- 1.0	+ 4.1	- .14	+ 1.46

The constant terms in these equations will be the same as those in the condition equations and the other elements in the first row are obtained as follows

$$+ 3.5 = 1.5\text{x}1\text{x}1 + 1.2\text{x}1\text{x}1 + 1.0\text{x}0\text{x}0 + .8\text{x}(-1)\text{x}(-1) + 1.6\text{x}0\text{x}0 + 1.6\text{x}0\text{x}0$$

$$- 1.2 = 1.5\text{x}1\text{x}0 + 1.2\text{x}1\text{x}(-1) + 1.0\text{x}0\text{x}1 + .8\text{x}(-1)\text{x}0 + 1.6\text{x}0\text{x}0 + 1.6\text{x}0\text{x}(-1)$$

$$- 1.5 = 1.5\text{x}1\text{x}(-1) + 1.2\text{x}1\text{x}0 + 1.0\text{x}0\text{x}(-1) + .8\text{x}(-1)\text{x}0 + 1.6\text{x}0\text{x}1 + 1.6\text{x}0\text{x}0$$

$$+ .79 = -.01 + 1.5\text{x}1\text{x}0 + 1.2\text{x}1\text{x}0 + 1.0\text{x}0\text{x}0 + .8\text{x}(-1)\text{x}(-1) + 1.6\text{x}0\text{x}1 + 1.6\text{x}0\text{x}1$$

Note that for the $\sum$ column, as well as multiplying the first row of the condition equations by the S row and by the weight, the constant d, must also be added in to give the check.

$$+ 3.5 - 1.2 - 1.5 - .01 = + .79$$

It is essential that the constant d, should be added in when obtaining $\sum$ as otherwise the constants in the subsequent solution of the equations would be unchecked

The Normal Equations shown above are those used in Appendix B3 as an example of the Gauss-Doolittle method of solution and the values of the Lagrangian multipliers, obtained there, are

$\ell_1 = +.0162 \qquad \ell_2 = -.0086 \qquad$ and $\ell_3 = .0380$

The corrections $v_1\ v_2\ \ldots\ldots\ v_6$ are now obtained from the Correlate equations e.g.

$$\begin{aligned}
v_1 &= 1.5(+.0162 - .0380) = -.0327\\
v_2 &= 1.2(+.0162 + .0086) = +.0298\\
v_3 &= 1.0(-.0086 - .0380) = -.0466\\
v_4 &= .8(-.0162) = -.0130\\
v_5 &= 1.6(\ .0380) = +.0608\\
v_6 &= 1.6(\ .0086) = +.0138
\end{aligned}$$

and it will be found that these corrections give the same values for the heights of B, C & D as were obtained by the parametric method.

7. Demonstration of the meaning effect of the Least Square Procedure

The same numerical example will again be used. In order to fix the heights of B, C & D from the known height of A, only three observations of differences of height are necessary, but six observations have actually been made. From these 6, 20 sets of 3 can be selected but not all will be suitable as if the 3 lines selected form a triangle, the height of at least one point will be unfixed. There are 4 triangles and hence the total number of possible sets which will give the required results is 16.

If any set of 3 observations are accepted as correct, this means that the corrections to the observed differences in height along these 3 lines are zero and the remaining corrections can then be obtained from the condition equations.

$$
\begin{array}{lllllll}
v_1 & + v_2 & & -v_4 & & & - .01 = 0 \\
 & - v_2 & + v_3 & & & -v_6 & + .09 = 0 \\
- v_1 & & - v_3 & & +v_5 & & - .14 = 0
\end{array}
$$

For instance if the observations along lines AB, AD & DC are accepted $v_1 = v_3 = v_6 = 0$ and from the above equations

$$v_2 = +.09 \qquad v_4 = +.08 \qquad \text{and} \quad v_5 = +.14$$

The weight of this set of corrections will be proportional to the product of the weights of the 3 selected observations, i.e. $\frac{1}{15} \cdot \frac{1}{10} \cdot \frac{1}{16}$ and in the following tabulation this has been multiplied by 12^3 in order to give figures near unity.

Data Accepted	v_1	v_2	v_3	v_4	v_5	v_6	12^3 x Weight
AB AC AD	0	0	0	-.01	+.14	+.09	.9600
AB AC BD	0	0	-.14	-.01	0	-.05	.6000
AB AC CD	0	0	-.09	-.01	+.05	0	.6000
AB AD BC	0	+.01	0	0	+.14	+.08	1.4400
AB AD DC	0	+.09	0	+.08	+.14	0	.7200
AB BC BD	0	+.01	-.14	0	0	-.06	.9000
AB BC CD	0	+.01	-.08	0	+.06	0	.9000
AB BD DC	0	-.05	-.14	-.06	0	0	.4500
AC AD CB	+.01	0	0	0	+.15	+.09	1.8000
AC AD DB	-.14	0	0	-.15	0	+.09	.9000
AC CD CB	+.01	0	-.09	0	+.06	0	1.1250
AC CD DB	-.05	0	-.09	-.06	0	0	.5625
AC CB BD	+.01	0	-.15	0	0	-.06	1.1250
AC DC DB	-.14	+.09	0	-.06	0	0	.6750
AD DC CB	-.08	+.09	0	0	+.06	0	1.3500
AD DB BC	-.14	+.15	0	0	0	-.06	1.3500
Weighted Means	-.0327	+.0297	-.0466	-.0130	+.0608	+.0137	15.4575

It will be seen that the weighted means are identical with the corrections obtained in Sections 6 & 5 except for minor rounding off errors.

8. Relationship between the Parametric and Condition Equation Methods

As has been previously stated the Condition Equations are merely those equations which are left after the parameters have been eliminated by some process from the Parametric Equations.

The Parametric Equations are $AX + C = V$

and the Condition Equations are $BV + D = 0$

$BAX + BC + D = B(AX + C) + D = BV + D = 0$ must be independent of X

$$\therefore\ BA = 0 \quad \text{and } BC + D = 0 \qquad 8.1$$

In our numerical example

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -.01 \\ +.14 \\ +.09 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} -.01 \\ +.09 \\ -.14 \end{bmatrix}$$

and matrix multiplication gives

$$BA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \;\&\; BC = \begin{bmatrix} +.01 \\ -.09 \\ +.14 \end{bmatrix} = -D$$

It should be noted that the A matrix is unique but the B matrix can take several different forms e.g. we could have had

$$B = \begin{bmatrix} 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} +.08 \\ -.09 \\ +.06 \end{bmatrix}$$

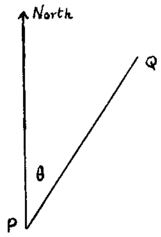
and the relationships $BA = 0$ and $BC + D = 0$ would still have been satisfied.

9. Procedure when the relationship between the required parameters and the observed quantities is not linear.

In the condition equation method, the geometric conditions which the adjusted observations have to satisfy can, as regards survey adjustments, often be expressed in simple form and they will not be affected by the fact that the relationship between the parameters and the adjusted observations is non linear.

This is not the case however with the Parametric Method of adjustment and, for this method, approximate values of the parameters must first be found and the relationship linearised by use of Taylor's Series expansions as far as the first order terms.

This assumes that the approximate values are such that the neglect of the second and higher order terms has no significant effect on the final results. For planimetric adjustment this method has been named the Variation of Co-ordinates Method.



Consider the observation firstly of the direction and secondly of the distance of a line PQ

Let the approximate co-ordinates of P be (E_p, N_p) and the exact ones $(E_p + \Delta E_p, N_p + \Delta N_p)$

Let the approximate co-ordinates of Q be (E_Q, N_Q) and the exact ones $(E_Q + \Delta E_Q, N_Q + \Delta N_Q)$

Let the approximate bearing of the line PQ be Θ and the exact bearing $(\Theta + \Delta\Theta)$

Let the approximate distance of the line PQ be ℓ and the exact distance $(\ell + \Delta\ell)$

The approximate bearing and distance will be computed from the approximate co-ordinates.

Finally let the measured direction be D, its correction v_D, the measured distance be d, its correction v_d and let O_p and $(O_p + \Delta O_p)$ be approximate and exact orienting factors for the observing station P, which are required to convert the measured direction into a true bearing.

Then considering first the direction observation

$$\text{True Bearing} = \Theta + \Delta\Theta = D + v_D + O_p + \Delta O_p \qquad 9.1$$

but $\log \tan (\Theta + \Delta\Theta) = \log (E_Q + \Delta E_Q - E_p - \Delta E_p) - \log (N_Q + \Delta N_Q - N_p - \Delta N_p)$

or $\log \tan \theta + \Delta\theta \dfrac{\sec^2\theta}{\tan \theta} \approx \log (E_Q - E_p) + \dfrac{(\Delta E_Q - \Delta E_p)}{E_Q - E_p} - \log (N_Q - N_p) - \dfrac{(\Delta N_Q - \Delta N_p)}{N_Q - N_p}$

but $\log \tan \theta = \log (E_Q - E_p) - \log (N_Q - N_p)$

and hence if $\Delta\theta$ is measured not in radians but in seconds

$$\Delta\theta'' \approx 206265 \frac{\sin 2\theta}{2} \left\{ \frac{\Delta E_Q - \Delta E_p}{E_Q - E_p} - \frac{\Delta N_Q - \Delta N_p}{N_Q - N_p} \right\}$$

The parametric equation for a direction observation thus becomes

$$v_D = 103132.5 \sin 2\theta \left\{ \frac{\Delta E_Q - \Delta E_p}{E_Q - E_p} - \frac{\Delta N_Q - \Delta N_p}{N_Q - N_p} \right\} - \Delta O_p + (\theta - D - O_p) \qquad 9.2$$

For a distance observation we have

$$\text{True distance} = \ell + \Delta\ell = d + v_d \qquad 9.3$$

but $\ell + \Delta\ell = \sqrt{(E_Q + \Delta E_Q - E_p - \Delta E_p)^2 + (N_Q + \Delta N_Q - N_p - \Delta N_p)^2}$

$$\ell + \Delta\ell \approx \sqrt{(E_Q - E_p)^2 + (N_Q - N_p)^2} + (\Delta E_Q - \Delta E_p)\frac{(E_Q - E_p)}{\ell} + (\Delta N_Q - \Delta N_p)\frac{(N_Q - N_p)}{\ell}$$

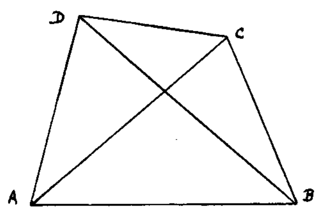
which reduces to $\Delta\ell = \sin \theta(\Delta E_Q - \Delta E_p) + \cos \theta(\Delta N_Q - \Delta N_p)$

and the parametric equation for a distance observation is

$$v_d = \sin \theta(\Delta E_Q - \Delta E_p) + \cos \theta(\Delta N_Q - \Delta N_p) + (\ell - d) \qquad 9.4$$

In this method there will be 2 co-ordinate parameters ΔE and ΔN for each new point to be fixed and one orienting parameter ΔO for every station (new or old) at which directions have been observed.

As an example let us derive in symbol form the parametric equations for a braced quadrilateral ABCD, fully observed by directions, for the fixation of two new stations C & D from two known stations A & B



In this example there will be 12 direction observations and 8 parameters, made up of 2 co-ordinate parameters for each of the two new stations C & D and 1 orientation parameter for each of the four

stations. In order to cut down the amount of writing in the tabulation below of the parametric equations let

$$a_{PQ} = \frac{103132.5 \sin 2\Theta_{PQ}}{E_Q - E_P} \qquad b_{PQ} = \frac{103125 \sin 2\Theta_{PQ}}{N_Q - N_P} \qquad C_{PQ} = \Theta_{PQ} - D_{PQ} - O_P$$

Line	ΔE_C	ΔN_C	ΔE_D	ΔN_D	ΔO_A	ΔO_B	ΔO_C	ΔO_D	Constant
AD	0	0	a_{AD}	$-b_{AD}$	-1	0	0	0	c_{AD}
AC	a_{AC}	$-b_{AC}$	0	0	-1	0	0	0	c_{AC}
AB	0	0	0	0	-1	0	0	0	c_{AB}
BA	0	0	0	0	0	-1	0	0	c_{BA}
BD	0	0	a_{BD}	$-b_{BD}$	0	-1	0	0	c_{BD}
BC	a_{BC}	$-b_{BC}$	0	0	0	-1	0	0	c_{BC}
CB	$-a_{CB}$	b_{CB}	0	0	0	0	-1	0	c_{CB}
CA	$-a_{CA}$	b_{CA}	0	0	0	0	-1	0	c_{CA}
CD	$-a_{CD}$	b_{CD}	a_{CD}	$-b_{CD}$	0	0	-1	0	c_{CD}
DC	a_{DC}	$-b_{DC}$	$-a_{DC}$	b_{DC}	0	0	0	-1	c_{DC}
DB	0	0	$-a_{DB}$	b_{DB}	0	0	0	-1	c_{DB}
DA	0	0	$-a_{DA}$	b_{DA}	0	0	0	-1	c_{DA}

It should be noted that if a line has been observed for direction at both ends, the coefficients of the co-ordinate parameters will be identical for both observations but the coefficients of the orientation parameters and the constant terms will be different. This can easily be seen to be the case as

$$a_{PQ} = \frac{103132.5 \sin 2\Theta_{PQ}}{E_Q - E_P} = \frac{-103132.5 \sin 2\Theta_{QP}}{E_P - E_Q} = -a_{QP}$$

since $\sin 2\Theta_{QP} = \sin 2(\Theta_{PQ} + 180) = \sin 2\Theta_{PQ}$

and similarly $b_{PQ} = -b_{QP}$

If the figure had been observed by trilateration instead of triangulation i.e. all the distances except AB had been measured then there would have been no orientation parameters and the parametric equations would have been

Line	ΔE_C	ΔN_C	ΔE_D	ΔN_D	Constant
AC	$\sin\Theta_{AC}$	$\cos\Theta_{AC}$	0	0	$\ell_{AC} - d_{AC}$
AD	0	0	$\sin\Theta_{AD}$	$\cos\Theta_{AD}$	$\ell_{AD} - d_{AD}$
BC	$\sin\Theta_{BC}$	$\cos\Theta_{BC}$	0	0	$\ell_{BC} - d_{BC}$
BD	0	0	$\sin\Theta_{BD}$	$\cos\Theta_{BD}$	$\ell_{BD} - d_{BD}$
CD	$-\sin\Theta_{CD}$	$-\cos\Theta_{CD}$	$\sin\Theta_{CD}$	$\cos\Theta_{CD}$	$\ell_{CD} - d_{CD}$

A combination of direction and distance observations can be dealt with similarly, the individual observations being weighted inversely to their variances.

The simplest example to illustrate the Variation of Co-ordinates method is the adjustment of the observations for a resected point, as irrespective of the number of rays observed there will be only 3 parameters, two for the co-ordinates of the resected point and one for the orientation at that point.

Numerical Example

Consider the fixation of a resected point with approximate co-ordinates (64908, 56627) given the following data

Station Observed	Easting	Northing	Theodolite Reading
Quartz	60060.660	59232.227	296° 28' 21.8"
Koppie	62589.399	61717.848	333 43 47.6
Corona	50019.962	36511.864	214 43 41.9
F.G.3	67379.350	63232.800	18 43 50.4
Knob	66140.580	58012.682	39 52 07.8

The first step is to calculate the approximate bearings

	Quartz	Koppie	Corona	F.G.3	Knob
E-Trig	60060.660	62589.399	50019.962	67379.350	66140.580
R P	64908.000	64908.000	64908.000	64908.000	64908.000
ΔE	-4847.340	-2318.601	-14888.038	+2471.350	+1232.580
N-Trig	59232.227	61717.848	36511.864	63232.800	58012.682
R P	56627.000	56627.000	56627.000	56627.000	56627.000
ΔN	+2605.227	+5090.848	-20115.136	+6605.800	+1385.682
Θ	$298^{\circ}15'21.7$	$335^{\circ}30'48.0$	$216^{\circ}30'24.0$	$20^{\circ}30'42.3''$	$41^{\circ}39'12.5''$
D	296 28 21.8	333 43 47.6	214 43 41.9	18 43 50.4	39 52 07.8
Θ-D	+1 46 59.9	+1 47 00.4	+1 46 42.1	+1 46 51.9	+1 47 04.7
O			1 46 42.1		
Θ-D-O	+17.8	+18.3	O	+9.8	+22.6
sin 2Θ	-.83400194	-.75440415	+.95637277	+.65636852	+.99318478

The approximate value O of the orientation correction can be chosen quite arbitrarily and in this case the value for the most distant station, Corona, has been accepted.

From the calculations given above and equations 9.2 the following tabulation of the Parametric Equations is obtained. All observations will be taken as of weight unity and in order to make the coefficients for ΔE and ΔN close to 1, the calculation will be carried out in units of centimetres and not of metres as shown above.

ΔE	ΔN	ΔO	Constant	Check
-.17744	-.33015	-1	+17.8	+16.29241
-.33556	-.15283	-1	+18.3	+16.81161
+.06625	-.04903	-1	O	- 0.98278
-.27391	+.10247	-1	+ 9.8	+ 8.62856
-.83102	+.73920	-1	+22.6	+21.50818

From these Parametric equations we get the following Normal Equations.

ΔE	ΔN	ΔO	Constant	Check
+.91410	-.53574	+1.55168	-30.76455	-28.83451
-.53574	+.69168	- .30966	+ 9.03667	+ 8.88294
+1.55168	-.30966	5.00000	-68.50000	-62.25798

These Normal Equations are those used in the example in Appendix B4 of the method of solving a set of equations by forming the inverse matrix and on reference to this example it will be found that the solution is

ΔE = +43.9 cms ΔN = +21.6 cms ΔO = +1.4"

and the adjusted co-ordinates of the Resected Point will be

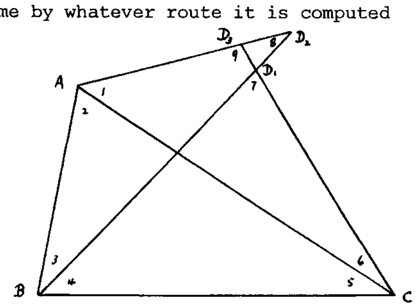
(64908.439, 56627.216)

10 Relative Merits of the Two Methods

For use on a programmable computer the Parametric Method is the more suitable as there is an easily formed parametric equation for each observation whereas in the Condition Equation Method a choice of conditions is possible and to programme the computer to ensure that the correct number of condition equations has been formed and that they are all independent is a matter of some complexity. In a planimetric survey network condition equations are of two types

(a) angle equations which (for a plane co-ordinate system) specify that the adjusted angles of a closed polygon of n sides must add up to (2n-4) right angles

(b) side equations which specify that the length of a side must be the same by whatever route it is computed



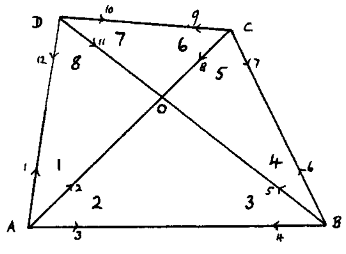
The figure above shows that since angle 9 = angle 7 + angle 8, all the triangular angle conditions could be satisfied but since the side equation is not satisfied there will be 3 different sets of co-ordinates for the point D

For a free net observed by triangulation only, Rainsford & Richardus (see bibliography) both give formulae for the number of angle equations and for the number of side equations, for both adjustment by directions and by angles, but the matter becomes more complicated if certain elements of the network are held fixed or if distances have been observed as well as directions.

If in addition to carrying out the adjustment, it is required to obtain estimates of the accuracies of the derived co-ordinates then, as will be shown later, the Parametric Method must be adopted as the condition equation method produces as its end result the corrections to the observations and not the final co-ordinates.

For small networks to be adjusted on a non-programmable machine, and without error analysis, the Condition Equation method may in some instances be the more suitable as it may entail a smaller amount of calculation. A certain amount of calculation is common to both systems and it is mainly the order of the normal equations which have to be solved which determines which method is preferable. The case of a braced quadrilateral, fully observed by directions, is a good example of an adjustment for which the condition equation method should be used. As shown in Section 9, by the Parametric Method there are twelve parametric equations in eight parameters and so a normal equation matrix of order 8. The number of condition equations is (12 - 8) = 4 and by this method there will be 4 normal equations only and hence much less calculation to be carried out. It will be shown moreover that by proper selection of the conditions the amount of calculation can be even further reduced.

Numerical Example



Data

Direction No.	Observed Plane Direction	Angle No.	Plane Angle
1	25° 39' 02.73"	1	29° 57' 57.70"
2	55 37 00.43	2	44 32 19.83
3	100 09 20.26	3	32 39 28.48
4	280 09 19.37	4	28 09 18.38
5	312 48 47.85	5	74 38 59.56
6	340 58 06.23	6	34 08 17.29
7	160 58 03.51	7	43 03 25.82
8	235 37 03.07	8	72 50 13.18
9	269 45 20.36		
10	89 45 21.02		
11	132 48 46.84		
12	205 39 00.02		

If v_i is the required correction to Direction No. i and w_i is the correction to Angle No. i, then with the numbering system shown in the diagram

$$w_{2i-1} = v_{3i-1} - v_{3i-2} \quad \& \quad w_{2i} = v_{3i} - v_{3i-1} \qquad i = 1 \text{ to } 4$$

A number of angular conditions are immediately obvious, for the whole quadrilateral and for the individual four triangles. These give the following conditions for the corrections

ABCD	$w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + w_7 + w_8 + 0.24 = 0$
ABC	$w_2 + w_3 + w_4 + w_5 + 6.25 = 0$
ABD	$w_1 + w_2 + w_3 + w_8 - 0.81 = 0$
BCD	$w_4 + w_5 + w_6 + w_7 + 1.05 = 0$
ACD	$w_1 + w_6 + w_7 + w_8 - 6.01 = 0$

Of these 5 conditions however only 3 are independent as

ABCD = ABC + ACD = ABD + BCD

and we can select any three of them provided they are independent. In this particular figure however a much better selection is made up of the three conditions

ABCD, (ABD - ABC) and (ABC - BCD)

as it will be seen later that these three conditions form an orthogonal system and so produce a purely diagonal section of the relevant part of the normal equation matrix.

We now have 3 angle conditions but a fourth condition is necessary and this will be a side equation. If O is the point of intersection of the diagonals of the quadrilateral, then

$$1 = \frac{OA}{OB}\cdot\frac{OB}{OC}\cdot\frac{OC}{OD}\cdot\frac{OD}{OA} = \frac{\sin(3+w_3)\sin(5+w_5)\sin(7+w_7)\sin(1+w_1)}{\sin(2+w_2)\sin(4+w_4)\sin(6+w_6)\sin(8+w_8)}$$

This is a non-linear relationship in the w's and it must be linearised before it can be used as a condition equation and the linearisation can be done in two ways

(a) <u>by a Taylors series expansion</u>

$$1 = \frac{\sin1\sin3\sin5\sin7}{\sin2\sin4\sin6\sin8}\left\{\begin{array}{r}1+w_1\cot1+w_3\cot3+w_5\cot5+w_7\cot7-w_2\cot2\\ -w_4\cot4-w_6\cot6-w_8\cot8\end{array}\right\}$$

or $w_1\cot1-w_2\cot2+w_3\cot3-w_4\cot4+w_5\cot5-w_6\cot6+w_7\cot7-w_8\cot8$

$$-206265\left\{\frac{\sin2\sin4\sin6\sin8}{\sin1\sin3\sin5\sin7}-1\right\} = 0$$

if the w's (and the v's) are measured in seconds of arc.

(b) <u>by taking logarithms</u>

$$\log\sin1+w_1D_1+\log\sin3+w_3D_3+\log\sin5+w_5D_5+\log\sin7+w_7D_7$$

$$= \log\sin2+w_2D_2+\log\sin4+w_4D_4+\log\sin6+w_6D_6+\log\sin8+w_8D_8$$

where D_i is the difference in the value of logsini for a change of 1", and the condition is then written as

$$w_1D_1-w_2D_2+w_3D_3-w_4D_4+w_5D_5-w_6D_6+w_7D_7-w_8D_8+(\log\sin1-\log\sin2+\log\sin3-\log\sin4+$$

$$\log\sin5-\log\sin6+\log\sin7-\log\sin8) = 0$$

The two equations express the same condition and it will be found that the coefficients and constant terms in the one equation are a constant multiple of those in the other.

It should be remarked that O need not be taken as the pole for deriving the side equation as any of the points of the quadrilateral could be taken instead e.g. for point A the condition would be

$$1 = \frac{AB}{AC}\cdot\frac{AC}{AD}\cdot\frac{AD}{AB} = \frac{\sin5}{\sin(3+4)}\cdot\frac{\sin(7+8)}{\sin6}\cdot\frac{\sin 3}{\sin8}$$

In the following figures, the Taylors series expansion method for pole O will be used together with the recommended angle conditions but instead of the w's we will work in terms of the v's as the original observations were directions. The condition equations and normal equations have been tabulated vertically merely because it was not possible to get in across the page all the figures required for the condition equations.

Condition and Correlate Equations					
	L_1	L_2	L_3	L_4	S
v_1	-1	-1		-1.734	-3.734
v_2		+1	-1	+2.750	+2.750
v_3	+1		+1	-1.016	+0.984
v_4	-1		-1	-1.560	-3.560
v_5		+1	+1	+3.429	+5.429
v_6	+1	-1		-1.869	-1.869
v_7	-1	+1		-0.275	-0.275
v_8		-1	+1	+1.750	+1.750
v_9	+1		-1	-1.475	-1.475
v_{10}	-1		+1	-1.070	-1.070
v_{11}		-1	-1	+1.379	-0.621
v_{12}	+1	+1		-0.309	+1.691
C	+0.24	-7.06	+5.20	-7.653	

Normal Equations				
L_1	+8	0	0	- 0.030
L_2	0	8	0	+ 6.069
L_3	0	0	8	+ 1.999
L_4	-0.030	+6.069	+ 1.999	+37.742
C	+0.240	-7.060	+ 5.200	- 7.653
$\sum$	+8.210	+7.009	+15.199	+38.127

The solution of this set of normal equations is

$L_1 = -.0296$ $L_2 = +.7989$ $L_3 = -.6775$ & $L_4 = .1102$

and substituting these values in the correlate equations gives the values of the corrections

$v_1 = -.959$ $v_2 = +1.776$ $v_3 = -.817$ $v_4 = +.537$ $v_5 = +.496$

$v_6 = -1.033$ $v_7 = +.800$ $v_8 = -1.287$ $v_9 = +.487$ $v_{10} = -.764$

$v_{11} = +.027$ $v_{12} = +.737$

On rounding off these figures to two decimal places and substituting in the angle condition equations it will be found that the first equation is satisfied exactly but that the second and third will total +.01 and -.01 respectively instead of zero. This suggests

an alteration to either v_2 or v_8 which occur with opposite signs in the two equations and all three equations will be satisfied if v_2 is rounded off to +1.77 instead of to +1.78

From these corrections the corrections to the angles can be calculated, the angles adjusted and finally the co-ordinates of the new stations found.

Error Analysis

11 Variances of Derived Quantities

When a survey adjustment has been carried out, we sometimes need to obtain an estimate of the precision of the final parameters, e.g. co-ordinates, heights etc. and these will of course be dependent on the precision of the original observations.

If M is a linear function of observed quantities x, y & z such that

$$M = ax + by + cz + d \qquad 11.1$$

where a, b, c & d are constants, then the variance of M is given by

$$\sigma_M^2 = a^2\sigma_x^2 + b^2\sigma_y^2 + c^2\sigma_z^2 + 2bc\sigma_{yz} + 2ca\sigma_{zx} + 2ab\sigma_{xy} \qquad 11.2$$

This last expression introduces, in addition to the variances σ_x^2, σ_y^2 & σ_z^2, three other functions, called co-variances, σ_{yz}, σ_{zx} and σ_{xy}. These co-variances constitute a measure of the correlation between the values of the quantities. If the measure of y is independent of that of x, then x & y are said to be uncorrelated and $\sigma_{xy} = 0$, but if the two measures are to a certain extent interdependent, as for instance the heights and weights of a group of individuals, then they are correlated and σ_{xy} will have a value.

Another measure of correlation is the coefficient of correlation ρ which is given by

$$\rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y} \qquad 11.3$$

and from statistics we know that $-1 \leq \rho \leq 1$

Other functions are now introduced.

The Variance Factor σ^2 which is a dimensionless quantity

The Weight Coefficients q_{xx} q_{xy} etc which are connected with the variances and co-variances by the relationships

$$\sigma_x^2 = \sigma^2 . q_{xx} \quad \text{and } \sigma_{xy} = \sigma^2 q_{xy}$$

The Weight Coefficients will thus have the same dimensions as the Variances and Covariances.

The Variance Operators q_x q_y etc which are merely mathematical operators, with no numerical values, which afford a simple method of deriving the weight coefficients from the relationships.

$$q_{xx} = q_x . q_x \quad \text{and } q_{xy} = q_x . q_y$$

Equation 11.2 can now be written in the form

$$\sigma_M^2 = \sigma^2 (aq_x + bq_y + cq_z)^2$$

and the method of derivation from equation 11.1 is obvious, the constant d having no variance.

If N is another linear function of x, y & z such that

$$N = rx + sy + tz + u$$

then $q_N = rq_x + sq_y + tq_z$

$$\begin{aligned} \text{and } \sigma_{MN} &= \sigma^2 (aq_x + bq_y + cq_z)(rq_x + sq_y + tq_z) \\ &= ar\sigma_x^2 + bs\sigma_y^2 + ct\sigma_z^2 + \sigma_{yz}(bt+cs) + \sigma_{zx}(cr+at) + \sigma_{xy}(as+br) \end{aligned}$$

Functions derived from observations, some of which are common to both functions, e.g. Northings and Eastings co-ordinates of a point, will thus be correlated.

The Law of Propagation of Variances, given above, is only valid if there is a linear relationship between the derived and observed quantities, and, if the relationship is not linear, it must first be linearised by a Taylor's series expansion.

$$\begin{aligned} M &= f(x, y, z) \\ &= f(x_o, y_o, z_o) + x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} + \text{second order terms} \end{aligned}$$

$$\text{and } q_M = \frac{\partial f}{\partial x} q_x + \frac{\partial f}{\partial y} q_y + \frac{\partial f}{\partial z} q_z \qquad 11.4$$

$$\sigma_M^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial f}{\partial z}\right)^2 \sigma_z^2 + 2\left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial f}{\partial z}\right)\sigma_{yz} + 2\left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial f}{\partial x}\right)\sigma_{zx} + 2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)\sigma_{xy} \tag{11.5}$$

As a very simple example of the use of these formulae, consider the following problem.

The sides of a rectangle have been measured as 150 and 100 units, both with a standard deviation of .01. It is assumed that opposite sides are exactly equal and that all angles are right angles. Calculate the variances of the Perimeter and of the Area and their co-variance.

Let the sides be x & y, the perimeter P and the area A Then P = 2(x + y) is a linear function of x & y and hence

$q_P = 2(q_x + q_y)$

$A = xy$ is not a linear function of x & y

$q_A = \frac{\partial A}{\partial x} q_x + \frac{\partial A}{\partial y} q_y = yq_x + xq_y$

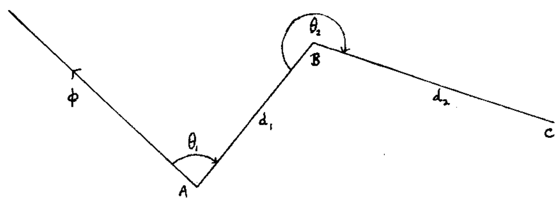
From this we obtain $\sigma_P^2 = 4(\sigma_x^2 + \sigma_y^2)$ since x & y are uncorrelated

$= 4(.0001+.0001) = .0008$

$\sigma_A^2 = y^2\sigma_x^2 + x^2\sigma_y^2 = 150^2(.01)^2 + 100^2(.01)^2 = 3.25$

$\sigma_{AP} = 2(y\sigma_x^2 + x\sigma_y^2) = 2(150 + 100)(.01)^2 = .05$

A further example will illustrate the importance of recognising when various functions are correlated



Consider a traverse run from A through B to C, the orienting ray at A having a bearing ϕ and the measured angles and distances being θ_1, θ_2, d_1 & d_2 as shown in the figure. To find the variance of the Easting co-ordinate of C in terms of that of A and of the observed quantities.

It will be assumed that the co-ordinates of A and the bearing ϕ are uncorrelated and the derivation will be carried out by two different methods

(a) $E_C = E_A + d_1\sin(\phi+\theta_1) + d_2\sin(\phi+\theta_1+\theta_2-180)$

$$= E_A + d_1\sin(\phi+\theta_1) - d_2\sin(\phi+\theta_1+\theta_2)$$

$$q_{E_C} = q_{E_A} + \sin(\phi+\theta_1)q_{d_1} - \sin(\phi+\theta_1+\theta_2)q_{d_2} + d_1\cos(\phi+\theta_1)(q_\phi+q_{\theta_1})$$
$$- d_2\cos(\phi+\theta_1+\theta_2)(q_\phi+q_{\theta_1}+q_{\theta_2})$$

Now it is obvious by common sense that none of the measures of θ_1 θ_2 d_1 & d_2 can be dependent on any of the others or on E_A and hence all the covariances will be zero and we need only consider the terms involving the variances

Squaring the last equation and multiplying by σ^2 gives

$$\sigma^2_{E_C} = \sigma^2_{E_A} + \sin^2(\phi+\theta_1)\sigma^2_{d_1} + \sin^2(\phi+\theta_1+\theta_2)\sigma^2_{d_2} + d_2^2\cos^2(\phi+\theta_1+\theta_2)\sigma^2_{\theta_2}$$
$$+ (d_1\cos(\phi+\theta_1) - d_2\cos(\phi+\theta_1+\theta_2))^2(\sigma^2_\phi + \sigma^2_{\theta_1})$$

(b) Instead of going direct from A to C we could first find the variance of E_B and then use that to find the variance of E_C and this method will bring in co-variances

$$E_B = E_A + d_1\sin(\phi+\theta_1)$$
$$q_{E_B} = q_{E_A} + \sin(\phi+\theta_1)q_{d_1} + d_1\cos(\phi+\theta_1)(q_\phi+q_{\theta_1})$$

Again the functions on the R.H.S. of the equation are uncorrelated and

$$\sigma^2_{E_B} = \sigma^2_{E_A} + \sin^2(\phi+\theta_1)\sigma^2_{d_1} + d_1^2\cos^2(\phi+\theta_1)(\sigma^2_\phi + \sigma^2_{\theta_1})$$
$$E_C = E_B - d_2\sin(\phi+\theta_1+\theta_2)$$
$$q_{E_C} = q_{E_B} - \sin(\phi+\theta_1+\theta_2)q_{d_2} - d_2\cos(\phi+\theta_1+\theta_2)(q_\phi+q_{\theta_1}+q_{\theta_2})$$

Now E_B will be correlated with θ_1 & ϕ and the remainder are uncorrelated

$$\sigma^2_{E_C} = \sigma^2_{E_B} + \sin^2(\phi+\theta_1+\theta_2)\sigma_{d_2} + d_2^2\cos^2(\phi+\theta_1+\theta_2)(\sigma^2_\phi + \sigma^2_{\theta_1} + \sigma^2_{\theta_2})$$
$$- 2d_2\cos(\phi+\theta_1+\theta_2)(\sigma_{E_B\phi} + \sigma_{E_B\theta_1})$$

but from the expression for q_{E_B} we obtain

$$\sigma_{E_B\phi} = d_1\cos(\phi+\theta_1)\sigma^2_\phi \text{ and } \sigma_{E_B\theta_1} = d_1\cos(\phi+\theta_1)\sigma^2_{\theta_1}$$

Substituting these values and also the expression for $\sigma^2_{E_B}$ gives

$$\sigma^2_{E_C} = \sigma^2_{E_A} + \sin^2(\phi+\Theta_1)\sigma^2_{d_1} + \sin^2(\phi+\Theta_1+\Theta_2)\sigma^2_{d_2} + d_2^2\cos^2(\phi+\Theta_1+\Theta_2)\sigma^2_{\Theta_2}$$

$$+(d_1\cos(\phi+\Theta_1)-d_2\cos(\phi+\Theta_1+\Theta_2))^2(\sigma^2_{\phi}+\sigma^2_{\Theta_1})$$

which is the same formula as was obtained by the previous method.

12 Extension to a large network

It is obvious that to proceed in this way to obtain extimates of the precision of the co-ordinates of points in a large network would entail a vast amount of calculation and so some simpler method of obtaining the required variances must be found. Using classical notation this is a cumbersome matter but it is relatively simple if matrix notation is used and this method will consequently be adopted.

In addition to the matrices and vectors used in Section 5 on the Parametric Method of adjustment, the following will now be needed.

Let Q_p be the vector of Variance Operators of the unadjusted observations (m,1)

Let Q_P be the vector of Variance Operators of the adjusted observations (m,1)

Let Q_X be the vector of Variance Operators of the parameters (n,1)

Let Q_{pp} be the matrix of Weight Coefficients of the unadjusted observations (m,m)

Let Q_{PP} be the matrix of Weight Coefficients of the adjusted observations (m,m)

Let Q_{XX} be the matrix of Weight Coefficients of the parameters (n,n)

and let the normal equation coefficient matrix $(A^TG^{-1}A)$ be denoted by N

The relationship between the matrices of Weight Coefficients and the vectors of Variance Operators is

$$Q_{XX} = Q_X.Q_X^T \qquad Q_{PP} = Q_P.Q_P^T$$

and $Q_{pp} = Q_p\,Q_p^T = G$ the diagonal matrix of Section 5

From equation 5.4, $X = -N^{-1}A^TG^{-1}C = -N^{-1}A^TG^{-1}(K-p)$

The only variable on the R.H.S. of this expression is p and hence

$$Q_X = N^{-1}A^TG^{-1}Q_p$$

and since both G and N are symmetrical matrices

$$Q_{XX} = Q_X . Q_X^T = N^{-1}A^TG^{-1}Q_pQ_p^TG^{-1}AN^{-1}$$

$$= N^{-1}A^TG^{-1}GG^{-1}AN^{-1} = N^{-1}A^TG^{-1}AN^{-1}$$

$$= N^{-1}NN^{-1} = N^{-1} \qquad 12.1$$

The matrix of Weight Coefficients of the Parameters is thus the inverse of the Normal Equation coefficient matrix and the variance-covariance matrix of the parameters $\left[\sigma_X\right]$ is given by

$$\left[\sigma_X\right] = \sigma^2 Q_{XX} = \sigma^2 N^{-1}$$

where if M is the minimum value of $V^TG^{-1}V$ an estimate of the variance factor $\sigma^2 = \frac{M}{m-n}$ (see Appendix 11 of a paper by V. Ashkenazi entitled "Adjustment of a control network for precise engineering surveys" in the Chartered Surveyor of January 1970 for a proof of this formula)

$$M = V^TG^{-1}V = (AX+C)^TG^{-1}(AX+C)$$

$$= X^TA^TG^{-1}(AX+C) + C^TG^{-1}V = C^TG^{-1}V \qquad 12.2$$

since $A^TG^{-1}(AX+C) = 0$ from the normal equations

$\therefore$ M can be calculated from $V^TG^{-1}V$ and checked from $C^TG^{-1}V$.

σ^2 the Variance Factor is a measure of how closely the variances of the observations correspond to those assumed in the G matrix used in the adjustment. A value of $\sigma^2 > 1$ indicates that the observations were less accurate than had been assumed and vice versa

13. Error Ellipses

The normal distribution of a single variable has a frequency distribution given by

$$f(x) = \frac{1}{\sigma_x\sqrt{2\pi}} e^{\frac{-(x-u)^2}{2\sigma_x^2}}$$

where u is the mean value of x and σ_x the standard deviation. For a multivariable distribution the joint frequency distribution function is, in matrix notation

$$f(x_1, x_2, \ldots\ x_n) = \frac{1}{(2\pi)^{\frac{n}{2}}(\det\left[\sigma_x\right])^{\frac{1}{2}}} e^{\frac{-(X-U)^T\left[\sigma_x\right]^{-1}(X-U)}{2}} \qquad 13.1$$

$$\text{where } \left[\sigma_x\right] = \begin{bmatrix} \sigma^2_{x_1} & \sigma_{x_1x_2} \cdots\cdots & \sigma_{x_1x_n} \\ \sigma_{x_1x_2} & \sigma^2_{x_2} \quad \cdots\cdots & \sigma_{x_2x_n} \\ & & \\ \sigma_{x_1x_n} & \sigma_{x_2x_n} \cdots\cdots & \sigma^2_{x_n} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} \quad U = \begin{bmatrix} u_1 \\ u_2 \\ \\ u_n \end{bmatrix}$$

u_j being again the mean value of the x_j distribution.

For the two dimensional case of planimetric co-ordinates with which we are dealing

$$\left[\sigma_x\right] = \begin{bmatrix} \sigma^2_{x_1} & \sigma_{x_1x_2} \\ \sigma_{x_1x_2} & \sigma^2_{x_2} \end{bmatrix} = \begin{bmatrix} \sigma^2_{x_1} & \rho\sigma_{x_1}\sigma x_2 \\ \rho\sigma_{x_1}\sigma_{x_2} & \sigma^2_{x_2} \end{bmatrix}$$

ρ being the coefficient of correlation.

$$\text{Determinant of } \left[\sigma_x\right] = \sigma^2_{x_1}\sigma^2_{x_2} - \rho^2\sigma^2_{x_1}\sigma^2_{x_2} = (1-\rho^2)\sigma^2_{x_1}\sigma^2_{x_2}$$

$$\left[\sigma_x\right]^{-1} = \frac{1}{1-\rho^2}\begin{bmatrix} 1/\sigma^2_{x_1} & -\rho/\sigma_{x_1}\sigma x_2 \\ -\rho/\sigma_{x_1}\sigma_{x_2} & 1/\sigma^2_{x_2} \end{bmatrix}$$

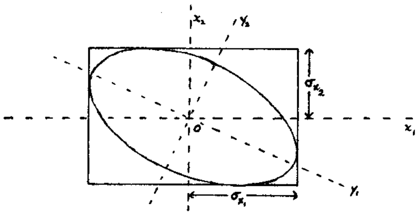
and a series of values $x_1\ x_2$ will have the same probability of occurring if $(X-U)^T \left[\sigma_x\right]^{-1}(X-U)$ is constant i.e.

$$\text{if } \left[x_1-u_1,\ x_2-u_2\right] \cdot \frac{1}{(1-\rho^2)}\begin{bmatrix} 1/\sigma^2_{x_1} & -\rho/\sigma_{x_1}\sigma_{x_2} \\ -\rho/\sigma_{x_1}\sigma_{x_2} & 1/\sigma^2_{x_2} \end{bmatrix} \cdot \begin{bmatrix} x_1-u_1 \\ x_2-u_2 \end{bmatrix} = \text{constant}$$

$$\text{or } \frac{(x_1-u_1)^2}{\sigma^2_{x_1}} - \frac{2\rho(x_1-u_1)(x_2-u_2)}{\sigma_{x_1}\sigma_{x_2}} + \frac{(x_2-u_2)^2}{\sigma^2_{x_2}} = \text{constant} \qquad 13.2$$

and this is the equation of an ellipse. If the constant = 1 the ellipse is called the standard error ellipse.

Thus corresponding to the standard deviation in one dimension we have an error ellipse in two dimensions and $\pm\sigma_{x_1}$ & $\pm\sigma_{x_2}$ will give its limits in the directions of the co-ordinate axes but the shape of the ellipse within these limits is not immediately known.



In order to find the size of the ellipse and its orientation it will be necessary to carry out a transformation to a new set of co-ordinate axes parallel to the major and minor axes of the ellipse. If co-ordinates relative to the new axes are y_1 and y_2 then the equation of the ellipse will be of the form

$$\frac{y_1^2}{\sigma_{y_1}^2} + \frac{y_2^2}{\sigma_{y_2}^2} = 1$$

i.e. with these axes ρ and $\sigma_{y_1y_2}$ will be zero and the lengths of the semi major and minor axes will be σ_{y_1} and σ_{y_2}

13.1 Derivation by the classical method of the lengths of the semi axes and the orientation of the major axis of the error ellipse

Let the x_1 axis be in the Easting direction in conformity with the practice of quoting Eastings before Northings, the y_1 axis be parallel to the major axis of the ellipse and let α be its bearing relative to the original axes.

The two sets of co-ordinates will be connected by the equations

$$y_1 = x_1\sin\alpha + x_2\cos\alpha$$

$$y_2 = -x_1\cos\alpha + x_2\sin\alpha$$

$\therefore q_{y_1} = \sin\alpha q_{x_1} + \cos\alpha q_{x_2}$ and $q_{y_2} = -\cos\alpha q_{x_1} + \sin\alpha q_{x_2}$

and the variances will be given by

$$\sigma_{y_1}^2 = \sin^2\alpha\sigma_{x_1}^2 + 2\sin\alpha\cos\alpha\sigma_{x_1x_2} + \cos^2\alpha\sigma_{x_2}^2 \qquad 13.1.1$$

$$\sigma_{y_2}^2 = \cos^2\alpha\sigma_{x_1}^2 - 2\cos\alpha\sin\alpha\sigma_{x_1x_2} + \sin^2\alpha\sigma_{x_2}^2 \qquad 13.1.2$$

$$\sigma_{y_1y_2} = \cos\alpha\sin\alpha(\sigma_{x_2}^2 - \sigma_{x_1}^2) - (\cos^2\alpha - \sin^2\alpha)\sigma_{x_1x_2} \qquad 13.1.3$$

Now if α is such that the y_1 & y_2 axes are parallel to the axes of the ellipse σ_{y_1} will be a maximum and this value of α will be given by

$$\frac{d(\sigma^2_{y_1})}{d\alpha} = 2 \sin\alpha\cos\alpha(\sigma^2_{x_1} - \sigma^2_{x_2}) + 2(\cos^2\alpha - \sin^2\alpha)\sigma_{x_1x_2} = 0$$

and it will be seen that this entails $\sigma_{y_1y_2} = 0$

The orientation α of the major axis is thus given by

$$\tan 2\alpha = \frac{2\sigma_{x_1x_2}}{\sigma^2_{x_2} - \sigma^2_{x_1}}$$

This will give two values for 2α, differing by 180° and hence will give the orientation of the major and minor axes but the angle 2α can be put in its proper quadrant by the rule, obtained from a consideration of the second differential, that $\sin 2\alpha$ and $\sigma_{x_1x_2}$ must have the same sign.

The lengths of the semi-axes are obtained by use of the fact that certain expressions are invariant on rotation of the co-ordinate axes. Adding 13.1.1 and 13.1.2 gives

$$\sigma^2_{y_1} + \sigma^2_{y_2} = \sigma^2_{x_1} + \sigma^2_{x_2} \qquad 13.1.4$$

$$\begin{aligned}\sigma^2_{y_1}\sigma^2_{y_2} - (\sigma_{y_1y_2})^2 &= (\sin^2\alpha\sigma^2_{x_1} + 2\sin\alpha\cos\alpha\sigma_{x_1x_2} + \cos^2\alpha\sigma^2_{x_2})(\cos^2\alpha\sigma^2_{x_1} - \\ &\quad 2\sin\alpha\cos\alpha\sigma_{x_1x_2} + \sin^2\alpha\sigma^2_{x_2}) - (\cos\alpha\sin\alpha(\sigma^2_{x_2} - \sigma^2_{x_1}) \\ &\quad - (\cos^2\alpha - \sin^2\alpha)\sigma_{x_1x_2})^2 \\ &= (\sigma_{x_1x_2})^2(-4\cos^2\alpha\sin^2\alpha - (\cos^2\alpha - \sin^2\alpha)^2) \\ &\quad + \sigma^2_{x_1}\sigma^2_{x_2}(\cos^4\alpha + \sin^4\alpha + 2\cos^2\alpha\sin^2\alpha) \\ &= \sigma^2_{x_1}\sigma^2_{x_2} - (\sigma_{x_1x_2})^2 \qquad 13.1.5\end{aligned}$$

but if $y_,$ & y_2 are axes parallel to those of the ellipse $\sigma_{y_1y_2} = 0$

$$\therefore \sigma_{y_1}^2 \sigma_{y_2}^2 = \sigma_{x_1}^2 \sigma_{x_2}^2 - (\sigma_{x_1x_2})^2$$

and $(\sigma_{y_1} \pm \sigma_{y_2})^2 = \sigma_{y_1}^2 + \sigma_{y_2}^2 \pm 2\sigma_{y_1y_2}$

$$= \sigma_{x_1}^2 + \sigma_{x_2}^2 \pm 2\sqrt{\sigma_{x_1}^2 \sigma_{x_2}^2 - (\sigma_{x_1x_2})^2}$$

$$\sigma_{y_i} = \frac{1}{2} \left(\sqrt{\sigma_{x_1}^2 + \sigma_{x_2}^2 + 2\sqrt{\sigma_{x_1}^2 \sigma_{x_2}^2 - (\sigma_{x_1x_2})^2}} + \sqrt{\sigma_{x_1}^2 + \sigma_{x_2}^2 - 2\sqrt{\sigma_{x_1}^2 \sigma_{x_2}^2 - (\sigma_{x_1x_2})^2}} \right)$$

For calculation purposes it is better to work out the variances from the formulae

$$\sigma_{y_1}^2 = \frac{1}{4} \left(2(\sigma_{x_1}^2 + \sigma_{y_2}^2) + 2\sqrt{(\sigma_{x_1} + \sigma_{x_2})^2 - 4(\sigma_{x_1}^2 \sigma_{x_2}^2 - (\sigma_{x_1x_2})^2)}\right)$$

$$= \frac{1}{2} \left(\sigma_{x_1}^2 + \sigma_{x_2}^2 + \sqrt{(\sigma_{x_1} - \sigma_{x_2})^2 + 4(\sigma_{x_1x_2})^2}\right) \qquad 13.1.6$$

and similarly $\sigma_{y_2}^2 = \frac{1}{2} \left(\sigma_{x_1}^2 + \sigma_{x_2}^2 - \sqrt{(\sigma_{x_1} - \sigma_{x_2})^2 + 4(\sigma_{x_1x_2})^2}\right)$ 13.1.7

Numerical Example

Consider the example of the adjustment of the observations for a Resected Point in Section 9, for which the inverse matrix was calculated in Appendix B4. This gives

$$\begin{bmatrix} q_{EE} & q_{EN} \\ q_{EN} & q_{NN} \end{bmatrix} = \begin{bmatrix} 6.88990 & 4.50420 \\ 4.50420 & 4.43157 \end{bmatrix}$$

and on substituting the values of the parameters in the parametric equations, the following values are obtained for the corrections

$v_1 = +1.477$ $v_2 = -1.140$ $v_3 = +.439$ $v_4 = -1.426$ & $v_5 = +.650$

$\therefore$ minimum value of $V^T G^{-1} V = \sum v^2 = 6.12983$

and it is checked against gross error by $\sum cv = 6.14380$

A perfect check between these two values will not be obtained owing to the different number of decimal figures in c and in v and the effect of rounding off errors.

Variance Factor = $\frac{6.12983}{5 - 3}$ = 3.065

and the variance-covariance matrix $\begin{bmatrix} \sigma_E^2 & \sigma_{EN} \\ \sigma_{EN} & \sigma_N^2 \end{bmatrix} = \begin{bmatrix} +21.118 & +13.805 \\ +13.805 & +13.583 \end{bmatrix}$

$$\tan 2\alpha = \frac{2 \times 13.805}{13.583-21.118} = -3.66423$$

$2\alpha = 105^\circ 16'$ since $\sin 2\alpha$ is +ve $\quad \therefore \alpha = 52^\circ 38'$

$$\sigma_{y_1}^2 = \tfrac{1}{2}(21.118+13.583 + \sqrt{7.535^2 + 4(13.805)^2}) = \tfrac{1}{2}(34.701+28.620)$$

$$= 31.660$$

$$\sigma_{y_2}^2 = \tfrac{1}{2}(34.701 - 28.620) = 3.041$$

the semi major axis of the ellipse has a bearing of $52^\circ 38'$ and a length of $\sqrt{31.660}$ = 5.6 cms and the semi minor axis is 1.7 cms

13.2 Matrix Method of obtaining the lengths of the semi axes and the orientation of the major axis

The equation of the error ellipse is

$$(X - U)^T \left[\sigma_X\right]^{-1} (X - U) = 1$$

where $\left[\sigma_X\right]$ is the variance-covariance matrix $\begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1 x_2} \\ \sigma_{x_1 x_2} & \sigma_{x_2}^2 \end{bmatrix}$

The procedure of the previous section is equivalent to a transformation to new axes through the centre of the ellipse as origin and such that the off diagonal terms of the new variance-covariance matrix become zero. This is done by putting Z = X - U and Y = MZ where M is an orthogonal matrix ($M^{-1} = M^T$)

$$Q_Z = Q_X \quad \text{and } Q_Y = MQ_Z$$

$$\left[\sigma_Z\right] = \left[\sigma_X\right] \quad \text{and} \quad \left[\sigma_Y\right] = M\left[\sigma_Z\right]M^T = M\left[\sigma_X\right] M^T$$

The equation of the error ellipse now becomes

$$Z^T \left[\sigma_X\right]^{-1} Z = 1$$

$$(M^{-1}Y)^T \left[\sigma_X\right]^{-1} (M^{-1}Y) = 1$$

$$Y^T (M \left[\sigma_X\right]^{-1} M^T) Y = 1$$

$$Y^T \left[\sigma_Y\right]^{-1} Y = 1$$

and M is chosen such that $\left[\sigma_Y\right] = \begin{bmatrix} \sigma^2_{Y_1} & 0 \\ 0 & \sigma^2_{Y_2} \end{bmatrix}$

The characteristic equation of the matrix σ_Y is

$$(\sigma^2_{Y_1} - \lambda)(\sigma^2_{Y_2} - \lambda) = 0$$

and its roots will thus be the squares of the semi-axes

$$\begin{aligned} \text{But } \left| \sigma_Y - \lambda I \right| &= \left| M \sigma_X M^T - \lambda MM^T \right| \\ &= \left| M(\sigma_X - \lambda I) M^T \right| \\ &= \left| M \right| . \left| \sigma_X - \lambda I \right| . \left| M^T \right| \\ &= \left| \sigma_X - \lambda I \right| \end{aligned}$$

i.e. the characteristic equation of a matrix is invariant on transformation by an orthogonal matrix

$\therefore$ the roots of the characteristic equation of $\left[\sigma_X\right]$ will also give the squares of the semi axes of the ellipse.

Let λ_1 and λ_2 be the roots of the characteristic equation of $\left[\sigma_X\right]$. Then corresponding to λ_i $(i = 1,2)$ will be a vector $X_i = \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix}$ such that $AX_i = \lambda_i X_i$

i.e. for $i = 1$

$$(\sigma^2_{x_{11}} - \lambda_1)x_{11} + \sigma_{x_{11}x_{12}} x_{12} = 0$$

$$\sigma_{x_{11}x_{12}} x_{11} + (\sigma^2_{x_{12}} - \lambda_1)x_{12} = 0$$

and the direction of the vector will be given by

$$\tan\alpha = \frac{x_{11}}{x_{12}} = \frac{\sigma_{x_{11}x_{12}}}{\lambda_1 - \sigma^2_{x_{11}}} = \frac{\lambda_1 - \sigma^2_{x_{12}}}{\sigma_{x_{11}x_{12}}}$$

Using the same numerical example as in the last section, the characteristic equation of the matrix $\left[\sigma_X\right] = \begin{bmatrix} 21.118 & 13.805 \\ 13.805 & 13.583 \end{bmatrix}$ is

$$(21.118 - \lambda)(13.583 - \lambda) - 13.805^2 = 0$$

$$\lambda^2 - 34.701\lambda + 96.268 = 0$$

$$\lambda = \frac{1}{2}(34.701 \pm \sqrt{34.701^2 - 4x96.268}) = 31.6605 \text{ or } 3.0405$$

and the semi axes will be 5.6 cms and 1.7 cms as before.
The orientation of the major axis will be given by

$$\tan\alpha = \frac{13.805}{31.6605 - 21.118} = 1.309$$

and $\alpha = 52^\circ 38'$

14. Relative Error Ellipses and Standard Deviation of the adjusted lengths and directions.

In some cases it may be more important to obtain estimates of the precision of the relative positions of two points rather than of their absolute positions. These estimates can also be found from the inverse of the Normal Equation matrix.
Consider two points A and B fixed in the adjustment of a triangulation network and for which the relative section of the variance co-variance matrix is

$$\sigma^2 N^{-1} = \sigma^2 \begin{bmatrix} q_{E_A E_A} & q_{E_A N_A} & q_{E_A E_B} & q_{E_A N_B} \\ & q_{N_A N_A} & q_{N_A E_B} & q_{N_A N_B} \\ & & q_{E_B E_B} & q_{E_B N_B} \\ & & & q_{N_B N_B} \end{bmatrix} = \begin{bmatrix} +8.91 & -2.74 & +0.52 & +0.72 \\ & +5.26 & -0.68 & +0.98 \\ & & +2.95 & +1.97 \\ & & & +3.28 \end{bmatrix}$$

For the relative error ellipse

$$\Delta E = E_B - E_A \qquad \Delta N = N_B - N_A$$

$$q_{\Delta E} = q_{E_B} - q_{E_A} \qquad q_{\Delta N} = q_{N_B} - q_{N_A}$$

$$\sigma^2_{\Delta E} = \sigma^2 q^2_{\Delta E} = \sigma^2_{E_B} - 2\sigma_{E_B E_A} + \sigma^2_{E_A}$$

$$= 2.95 - 2(0.52) + 8.91 = 10.82$$

$$\sigma^2_{\Delta N} = \sigma^2_{N_B} - 2\sigma_{N_B N_A} + \sigma^2_{N_A} = 3.28 - 2(0.98) + 5.26 = +6.58$$

$$\sigma_{\Delta E \Delta N} = \sigma_{N_B E_B} + \sigma_{N_A E_A} - \sigma_{N_B E_A} - \sigma_{N_A E_B}$$

$$= 1.97 - 2.74 - 0.72 + 0.68 = -.81$$

and the lengths of the semi axes and the orientation of the major axis can then be obtained by consideration of the characteristic equation of the matrix $\begin{bmatrix} 10.82 & -.81 \\ -.81 & 6.58 \end{bmatrix}$

The standard deviations of the distance and direction of AB can also be found by rotation of the co-ordinate axes so that they lie along AB and perpendicular to it. Assume that the adjusted length AB = 325000 and the bearing 160°27'00". Then by substituting in equation 13.1.1 and 13.1.2 with

$\alpha = 160°27'00"$, $\sigma^2_{x_1} = 10.82$ etc. we obtain

Variance of distance $= \sigma^2_{y_1} = 7.5656$

and 325000^2 (variance of direction) $= \sigma^2_{y_2} = 9.8345$

Standard deviation of distance $= \sqrt{7.5656} = 2.75$

Standard deviation of direction (in seconds of arc)

$$= \frac{206265}{325000} \sqrt{9.8345} = 2"$$

Appendix A Matrix Theory Revision

A1. This appendix deals only with those aspects of Matrix Theory, a knowledge of which is essential for the understanding of the development by matrix methods of the theory of Least Squares.

An arrangement of symbols or numbers in rows and columns to form rectangles is called an array. If it has m rows and n columns it is said to be of order (m x n), the number of rows always being stated first.

A matrix is an array which obeys certain rules of combination with other matrices. It will be shown in full by putting square brackets round the individual elements of the matrix e.g. $\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$ and can also be denoted by a single letter, in this case X. It should be noted that the elements of the matrix given above are x_{ij} where i is the number of the row and j is the number of the column in which the element occurs.

A2. Rules that matrices must obey

(i) Addition. This is defined only for matrices of the same order. If X & Y are two matrices both of order (m x n) then the elements of another (m x n) matrix Z which equals X + Y are given by $z_{ij} = x_{ij} + y_{ij}$

It is easy to see that matrix addition is commutative, i.e.

$$X + Y = Y + X$$

Matrix addition is also associative

$$(X + Y) + Z = X + (Y + Z) = X + Y + Z$$

(ii) Multiplication by a scalar

A scalar is a single number (or symbol representing a constant number). The multiplication of a matrix X by a scalar λ is formed by multiplying each element x_{ij} of X by the constant λ

It follows that $(\lambda_1 + \lambda_2)X = \lambda_1 X + \lambda_2 X$

and $\lambda(X + Y) = \lambda X + \lambda Y$

(iii) Subtraction of Matrices

By putting $\lambda = -1$ and using the results given above the differences of two matrices of the same order can be defined.

e.g. $Z = X + (-1)Y = X - Y$ where $z_{ij} = x_{ij} - y_{ij}$

(iv) Multiplication of Matrices

Two matrices can only be multiplied together if the left hand matrix has the same number of columns as the right hand matrix has rows. The product will be a matrix with the same number of rows as the left hand matrix and the same number of columns as the right hand matrix.

$$\underset{m \times p}{Z} = \underset{m \times n}{X} \; \underset{n \times p}{Y}$$

and the elements of the product matrix Z are obtained by multiplying all the elements of one row of X by the corresponding elements of one column of Y and summing the results.

$$z_{ij} = x_{i1}y_{1j} + x_{i2}y_{2j} + \ldots + x_{in}y_{nj} = \sum_{k=1}^{n} x_{ik}y_{kj}$$

It is evident from this definition that matrix multiplication is in general not commutative as in the instance given above $\underset{n \times p}{Y} \cdot \underset{m \times n}{X}$ would have no meaning since $p \neq m$. If $p = m$ the same statement holds good since $\underset{m \times n}{X} \cdot \underset{n \times m}{Y}$ would give as a product an $(m \times m)$ matrix whilst $\underset{n \times m}{Y} \cdot \underset{m \times n}{X}$ would give an $(n \times n)$ matrix.

Even if $n = m$, matrix multiplication of two matrices X & Y of order $(m \times m)$ will still in general not be commutative though there will be special cases when XY will equal YX

A3. Special Types of Matrices

Vectors are matrices with either a single row (row vector) or a single column (column vector)

Scalars can be regarded as (1×1) matrices

Square Matrix is a matrix which has the same number of rows and columns. For a square matrix the diagonal from the top left hand corner to the bottom right hand corner, i.e. that containing the elements a_{ii}, is called the main diagonal

Diagonal Matrix is one in which all the elements except those on the main diagonal are zero.

Unit Matrix is a diagonal matrix in which all the diagonal elements are equal to 1.

Upper Triangular Matrix is a matrix in which all the elements below the main diagonal are zero.

Lower Triangular Matrix is a matrix in which all the elements above the main diagonal are zero

A4. Other Definitions

The Transpose of a matrix X, denoted by X^T, is obtained by interchanging the rows and columns of the original matrix X

$$(x^T)_{ij} = x_{ji}$$

It will be seen that if X is an (m x n) matrix, X^T will be an (n x m) matrix.

The Inverse of a matrix X will be defined only for square matrices and will be denoted by X^{-1} such that

$$XX^{-1} = X^{-1}X = I \qquad \text{A4.1}$$

where I is the unit matrix of the same order as X. The inverse will only exist if the determinant of the matrix X, denoted by $|X|$ is not zero. If $|X| = 0$ the matrix X is said to be SINGULAR. A matrix and its inverse are one of the exceptions for which multiplication is commutative.

An Orthogonal Matrix is one for which

(a) the sum of the squares of the elements in any row or column = 1

$$\sum_{i=1}^{n} a_{ij}^2 = \sum_{j=1}^{n} a_{ij}^2 = 1$$

and (b) the sum of the corresponding mixed products of any two rows or any two columns = 0

$$\sum_{i=1}^{n} a_{ij}a_{ik} = \sum_{j=1}^{n} a_{ij}a_{kj} = 0$$

or alternatively it can be defined by the relationship that its inverse and transpose are identical $A^{-1} = A^T$ A4.2

The determinant of an orthogonal matrix is ± 1

<u>The Rules for the Transpose or Inverse of products of matrices</u> are as follows

$$(ABC)^T = C^T B^T A^T$$

$$(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$$

i.e. the transpose or inverse of a product of matrices is equal to the product of the individual transposed or inverted matrices but taken in the reverse order.

A5 Latent Roots and Vectors

Given a square matrix A, there is at least one vector X and a corresponding scalar λ such that

$$A \cdot X = \lambda X \qquad \text{A5.1}$$

λ is called the latent root or eigenvalue of the matrix A

X is called the latent vector or eigenvector of the matrix A

The equation A5.1 can be written as $AX = \lambda IX$

or as $(A - \lambda I)X = 0$ A5.2

This is a set of homogeneous linear equations for which a non-trivial solution will only exist if the determinant

$$|A - \lambda I| = 0 \qquad \text{A5.3}$$

This relationship is called the Characteristic Equation of the matrix A and gives the values of λ and hence of X. The results of the solution of the characteristic equation can be expressed as

$$AY = YD \qquad \text{A5.4}$$

where Y is a matrix, the columns of which are the eigenvectors

D is a diagonal matrix, the elements of which are the corresponding eigenvalues

Let $A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$ then $|A - \lambda I| = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = 0$

or $(5-\lambda)(2-\lambda) - 4 = 0$

$$\lambda^2 - 7\lambda + 6 = 0 \text{ giving } \lambda = 1 \text{ or } 6$$

If $\lambda = 1$ $\quad 5x_1 + 2x_2 = x_1$ giving $X = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$2x_1 + 2x_2 = x_2$

If $\lambda = 6$ $\quad 5x_1 + 2x_2 = 6x_1$ giving $X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\quad 2x_1 + 2x_2 = 6x_2$

and it can easily be verified that $\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \times \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$

A6 Similarity Transformations

Pre- and post-multiplication of a matrix A by matrices H^{-1} and H respectively, produces a new matrix B, the latent roots of which are the same as those of A and whose latent vectors are $H^{-1}X$

$$\text{If} \quad AX = \lambda X$$

$$B(H^{-1}X) = (H^{-1}AH)(H^{-1}X) = H^{-1}AX$$

$$= H^{-1}\lambda X = \lambda(H^{-1}X)$$

A7 Bilinear and Quadratic Forms and their first derivative

$$A(x,y) = \begin{matrix} a_{11}x_1y_1 & + a_{12}x_1y_2 & + \cdots\cdots\cdots & + a_{1n}x_1y_n \\ + a_{21}x_2y_1 & + a_{22}x_2y_2 & + \cdots\cdots\cdots & + a_{2n}x_2y_n \\ \cdot\ \ \cdot & \cdot\ \ \cdot & & \cdot\ \ \cdot \\ \cdot\ \ \cdot & \cdot\ \ \cdot & & \cdot\ \ \cdot \\ + a_{m1}x_my_1 & + a_{m2}x_my_2 & + \cdots\cdots\cdots & + a_{mn}x_my_n \end{matrix} = X^TAY \qquad \text{A7.1}$$

where A is an (m x n) matrix, X an (m x 1) vector and Y an (n x 1) vector, is a <u>scalar</u> and is called the <u>Bilinear Form</u>

If X = Y then A(x,x) is called a <u>Quadratic Form</u> and in this case A must be a square matrix.

The bilinear form can be written as

$$F = A(x,y) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} y_j \qquad F = X^TAY = Y^TA^TX \qquad \text{A7.2}$$

Then differentiating with respect to y

$$dF = \sum_{i=1}^{m} x_i a_{ij} \sum_{j=1}^{n} dy_j \qquad dF = X^TA(dY) = (dY)^TA^TX \qquad \text{A7.3}$$

and differentiating with respect to x

$$dF = \sum_{i=1}^{m} dx_i \sum_{j=1}^{n} a_{ij} y_j \qquad dF = (dX)^TAY = Y^TA^T(dX) \qquad \text{A7.4}$$

Using the forms that express the differential as a column vector these results give $\dfrac{\partial (X^TAY)}{\partial X} = AY$ and $\dfrac{\partial (X^TAY)}{\partial Y} = A^TX$ $\qquad$ A7.5

If $Y = X$ and A is a square matrix then for the quadratic form

$$X^T A X$$

$$\frac{d}{dX}(X^T A X) = AX + A^T X \qquad \text{A 7.6}$$

and if A is a symmetrical matrix ($A^T = A$)

$$\frac{d}{dX}(X^T A X) = 2AX \qquad \text{A7.7}$$

A8 Positive Definite Matrices

If the quadratic form $X^T A X$ of a real and symmetrical square matrix A is always positive for any real and non-zero vector X then the matrix A is said to be POSITIVE DEFINITE

Positive Definite Matrices have a number of properties of which the most important for the purpose of their use in the theory of Least Squares is that all the eigenvalues are real and positive.

Appendix B Methods of Solving Sets of Linear Equations

B1 In the Theory of Least Squares it has been found that the Normal Equations formed in both the Parametric Equation method and the Condition Equation method are symmetrical and because of this special methods of solution, which are not of general application, can be adopted.

The most efficient methods of solution of linear equations, in terms of the number of numerical processes to be carried out, are the elimination methods in which the various parameters are eliminated one at a time. There are a number of different elimination methods of solution but the two most frequently used in survey adjustments are the Cholesky Method and the Gauss-Doolittle method.

These will now be described.

B2 The Cholesky Method

Let the normal equations be $NX + C = 0$ B2.1

The Cholesky method replaces this set of equations by another set

$$BX + K = 0 \qquad \text{B2.2}$$

which have the same roots and for which B is an upper triangular matrix chosen in such a way that $B^T B = N$ B2.3

$$(B^T)^{-1}\ (NX + C) = 0 \qquad \text{from B2.1}$$

$$(B^T)^{-1}\ B^T BX + (B^T)^{-1}C = 0$$

$$BX + (B^T)^{-1}C = 0 \qquad K = (B^T)^{-1}C$$

Hence the new constant terms K are obtained from the old ones C by the same processes which give the new coefficient matrix B from the old one N

The procedure is worked out, in full, below for a (3 x 3) matrix, a column for the constant terms and a column for checking purposes being added to N and to B (but not to B^T)

$$\begin{bmatrix} n_{11} & n_{12} & n_{13} & c_1 & s_1 \\ n_{12} & n_{22} & n_{23} & c_2 & s_2 \\ n_{13} & n_{23} & n_{33} & c_3 & s_3 \end{bmatrix} = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{12} & b_{22} & 0 \\ b_{13} & b_{23} & b_{33} \end{bmatrix} x \begin{bmatrix} b_{11} & b_{12} & b_{13} & k_1 & S_1 \\ 0 & b_{22} & b_{23} & k_2 & S_2 \\ 0 & 0 & b_{33} & k_3 & S_3 \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}^2 & b_{11}b_{12} & b_{11}b_{13} & b_{11}k_1 & b_{11}S_1 \\ b_{11}b_{12} & b_{12}^2+b_{22}^2 & b_{12}b_{13}+b_{22}b_{23} & b_{12}k_1+b_{22}k_2 & b_{12}S_1+b_{22}S_2 \\ b_{11}b_{13} & b_{12}b_{13}+b_{22}b_{23} & b_{13}^2+b_{23}^2+b_{33}^2 & b_{13}k_1+b_{23}k_1+b_{33}k_3 & b_{13}S_1+b_{23}S_2+b_{33}S_3 \end{bmatrix}$$

Equating the corresponding elements of the two matrices

$$b_{11} = \sqrt{n_{11}} \qquad b_{12} = \frac{n_{12}}{b_{11}} \qquad b_{13} = \frac{n_{13}}{b_{11}} \qquad k_1 = \frac{c_1}{b_{11}} \qquad S_1 = \frac{s_1}{b_{11}}$$

$$b_{22} = \sqrt{n_{22} - b_{12}^2} \qquad b_{23} = \frac{n_{23} - b_{12}b_{13}}{b_{22}} \qquad k_2 = \frac{c_2 - b_{12}k_1}{b_{22}} \qquad S_2 = \frac{s_2 - b_{12}S_1}{b_{22}}$$

$$b_{33} = \sqrt{n_{33} - b_{13}^2 - b_{23}^2} \qquad k_3 = \frac{c_3 - b_{13}k_1 - b_{23}k_2}{b_{33}} \qquad S_3 = \frac{s_3 - b_{13}S_1 - b_{23}S_2}{b_{33}}$$

or in general $b_{ii} = \sqrt{n_{ii} - \left[b_{mi}^2\right]_{m=1}^{i-1}}$ B2.4

$$b_{ij} = \frac{n_{ij} - \left[b_{mi}b_{mj}\right]_{m=1}^{i-1}}{b_{ii}} \qquad j > i \qquad \text{B2.5}$$

$$k_i = \frac{c_i - \left[b_{mi}k_m\right]_{m=1}^{i-1}}{b_{ii}} \qquad \text{B2.6}$$

$$S_i = \frac{s_i - \left[b_{mi}S_m\right]_{m=1}^{i-1}}{b_{ii}} \qquad \text{B2.7}$$

<u>Numerical Example</u>

As an example let us take the normal equations from the Parametric Method example in Section 5

$$N = \begin{bmatrix} 3.05 & -1.5 & -.75 \\ -1.5 & 3.25 & -.75 \\ -.75 & -.75 & +2.70 \end{bmatrix} \quad C = \begin{bmatrix} +.0900 \\ +.0825 \\ -.1725 \end{bmatrix} \quad S = \begin{bmatrix} +.8900 \\ +1.0825 \\ +1.0275 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

x_1	x_2	x_3	k	s'
1.7464	- .8589	- .4295	+.0515	+ .5096
	+1.5850	- .7059	+.0800	+ .9591
		+1.4203	-.0661	+1.3542
- .0327	- .0298	+ .0465		
-1.0327	-1.0298	- .9535		

The figures in the top part of this schedule are obtained by application of formulae B2.4 to B2.7. For example the entries in the

second row are found as follows

$$1.5850 = \sqrt{3.25 - (-.8589)^2}$$

$$-.7059 = (-.75 - (-.8589(-.4295))/1.5850$$

$$+.0800 = (+.0825 - (-.8589)(+.0515))/1.5850$$

$$+.9591 = ((1.0825)-(-.8589)(+.5096))/1.5850$$

Each row should be checked by $S_i = k_i + b_{ii} + b_{i\,i+1} + \ldots b_{in}$ before proceeding to the calculation of the next row and any discrepancies greater than 1 or 2 in the last place of decimals (which might be due to rounding off errors) should be investigated and the errors rectified.

The top line in the bottom part of the tabulation gives the values of $x_{\wedge}$ x_2 and x_3 as found by back substitution and the bottom line gives a checking value obtained by considering the S' column as the constants of the equation instead of the k column.

If x_1, x_2, x_3 are the solutions using the correct constants k and X_1, X_2, X_3 are the solutions using the auxiliary constants S, then the top line of the tabulation gives two equations

$$b_{11}x_1 + b_{12}x_2 + b_{13}x_3 + k_1 = 0$$

$$b_{11}X_1 + b_{12}X_2 + b_{13}X_3 + S_1 = 0$$

but $S_1 = b_{11} + b_{12} + b_{13} + k_1$ and hence the second equation can be written as

$$b_{11}(X_1 + 1) + b_{12}(X_2 + 1) + b_{13}(X_3 + 1) + k_1 = 0$$

This is true of the other lines in the tabulation and hence

$$X_1 = x_1 - 1 \qquad X_2 = x_2 - 1 \qquad X_3 = x_3 - 1$$

thus providing a check on the back substitution. The figures for x_2 and X_2 are obtained as follows

$$-.0298 = -(.0800 + (-.7059)(.0465))/1.5850$$

$$-1.0298 = -(.9591 + (-.7059)(-.9535))/1.5850$$

Starting with the last equation the values of x_i and X_i are calculated as above and each set must be made to check by the relationship $X_i = x_i - 1$ before passing on to the calculation of the next value.

B.3 The Gauss-Doolittle Method

In this method also the original normal equations $NX + C = 0$ are replaced by a triangular system $BX + K = 0$ but in this case B is selected so that $N = B^T DB$ where B as before, is an upper triangular matrix and D is a diagonal matrix, the elements of which are the inverses of the diagonal elements of B

Again taking a (3 x 3) matrix to illustrate the procedure

$$\begin{bmatrix} n_{11} & n_{12} & n_{13} & c_1 & s_1 \\ n_{12} & n_{22} & n_{23} & c_2 & s_2 \\ n_{13} & n_{23} & n_{33} & c_3 & s_3 \end{bmatrix} = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{12} & b_{22} & 0 \\ b_{13} & b_{23} & b_{33} \end{bmatrix} \times \begin{bmatrix} 1/b_{11} & 0 & 0 \\ 0 & 1/b_{22} & 0 \\ 0 & 0 & 1/b_{33} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & b_{13} & k_1 & S_1 \\ 0 & b_{22} & b_{23} & k_2 & S_2 \\ 0 & 0 & b_{33} & k_3 & S_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ b_{12}/b_{11} & 1 & 0 \\ b_{13}/b_{11} & b_{23}/b_{22} & 1 \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & b_{13} & k_1 & S_1 \\ 0 & b_{22} & b_{23} & k_2 & S_2 \\ 0 & 0 & b_{33} & k_3 & S_3 \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} & k_1 & S_1 \\ b_{12} & \frac{b_{12}^2}{b_{11}} + b_{22} & \frac{b_{12}}{b_{11}}b_{13} + b_{23} & \frac{b_{12}}{b_{11}}k_1 + k_2 & \frac{b_{12}S_1}{b_{11}} + S_2 \\ b_{13} & \frac{b_{12}b_{13}}{b_{11}}+b_{23} & \frac{b_{13}^2}{b_{11}} + \frac{b_{23}^2}{b_{22}} + b_{33} & \frac{b_{13}}{b_{11}}k_1 + \frac{b_{23}}{b_{22}}k_2 + k_3 & \frac{b_{13}S_1}{b_{11}} + \frac{b_{23}S_2}{b_{22}} + S_3 \end{bmatrix}$$

Again equating corresponding elements of the two matrices

$b_{11} = n_{11}$ $\quad b_{12} = n_{12}$ $\quad b_{13} = n_{13}$ $\quad k_1 = c_1$ $\quad S_1 = s_1$

$b_{22} = n_{22} - \frac{b_{12}^2}{b_{11}}$ $\quad b_{23} = n_{23} - \frac{b_{12}b_{13}}{b_{11}}$ $\quad k_2 = c_2 - \frac{b_{12}k_1}{b_{11}}$ $\quad S_2 = s_2 - \frac{b_{12}S_1}{b_{11}}$

$b_{33} = n_{33} - \frac{b_{13}^2}{b_{11}} - \frac{b_{23}^2}{b_{22}}$ $\quad k_3 = c_3 - \frac{b_{13}k_1}{b_{11}} - \frac{b_{23}k_2}{b_{22}}$ $\quad S_3 = s_3 - \frac{b_{13}S_1}{b_{11}} - \frac{b_{23}S_2}{b_{22}}$

or in general $b_{ii} = n_{ii} - \left[\frac{b_{mi}^2}{b_{mm}}\right]_{m=1}^{i-1}$ B3.1

$$b_{ij} = n_{ij} - \left[\frac{b_{mi}b_{mj}}{b_{mm}}\right]_{m=1}^{i-1} \qquad \text{B3.2}$$

$$k_i = c_i - \left[\frac{b_{mi}k_m}{b_{mm}}\right]_{m=1}^{i-1} \qquad \text{B3.3}$$

$$S_i = s_i - \left[\frac{b_{mi}S_m}{b_{mm}}\right]_{m=1}^{i-1} \qquad \text{B3.4}$$

This time we will take for our numerical example the Normal Equations of the example of the Condition Equation method in Section 6

$$N = \begin{bmatrix} 3.5 & -1.2 & -1.5 \\ -1.2 & 3.8 & -1.0 \\ -1.5 & -1.0 & 4.1 \end{bmatrix} \quad C = \begin{bmatrix} -.01 \\ +.09 \\ -.14 \end{bmatrix} \quad S = \begin{bmatrix} +.79 \\ +1.69 \\ +1.46 \end{bmatrix}$$

The solution is tabulated as follows

B			k	S'
ℓ_1	ℓ_2	ℓ_3		
3.5	- 1.2	- 1.5	- .01	+ .79
- 1	+ .3429	+ .4286	+ .0029	- .2257
	+ 3.3886	- 1.5143	+ .0866	+ 1.9609
	- 1	+ .4469	- .0256	- .5787
		2.7804	- .1056	2.6748
		- 1	+ .0380	- .9620
+ .0162	- .0086	+ .0380		
- .9838	- 1.0086	- .9620		

The first row in the solution is the same as the first row in the normal equations and the second row is obtained by dividing throughout by the left hand term and changing the sign. Thus $+ .3429 = -b_{12}/b_{11}$ and $+.0029 = - k_1/b_{11}$. Again the check of the S' column against the sum of the other entries in the same row must be applied before proceeding to the calculation of the next row.

The figures in the third row are obtained as follows

$3.3886 = 3.8 + (- 1.2).(+ .3429)$

$- 1.5143 = -1.0 + (-1.2)(+ .4286) = - 1.0 + (.3429)(- 1.5)$

$+ .0866 = + .09 + (- 1.2)(+ .0029) = + .09 + (.3429)(- .01)$

$+ 1.9609 = + 1.69 + (- 1.2)(- .2257) = + 1.69 + (.3429)(+ .79)$

It should be noted that except for the left hand entry, each additive term can be calculated in two ways as for instance

$$-\frac{b_{mi}b_{mj}}{b_{mm}} = b_{mi}\left(\frac{-b_{mj}}{b_{mm}}\right) = b_{mj}\left(\frac{-b_{mi}}{b_{mm}}\right)$$

and the use of the expression which involves the two numbers closest together in magnitude will reduce rounding off errors.

The odd numbered rows are thus obtained by substituting in formulae B3.1 to B3.4 whilst the even numbered rows are obtained by dividing each element of the row immediately above by its left hand member and changing the sign.

In the back substitution only the even numbered rows are used and again a check is made by using the S' column as an auxiliary set of constants

e.g. $\ell_2 = (.4469)(.0380) - .0256 = -.0086$

$L_2 = (.4469)(-.9620) - .5787 = -1.0086$

B.4 Solution by calculating the Inverse Matrix

This is not an elimination method and it involves more mathematical operations than the previous methods do. If error analysis of the final results of a survey adjustment is to be carried out then the inverse of the normal equation matrix has to be calculated. Whilst, as will be shown later, the elimination methods can be adapted to give the inverse of the normal equation matrix, this involves many more calculations and so the methods lose much of their advantage.

With the same normal equations $NX + C = 0$, premultiplication by N^{-1} will change N into I the unit matrix, I into N^{-1} and C into -X where X is the solution, or adding a checking term S

$$N^{-1}\left[N \mid I \mid C \mid S\right] = \left[I \mid N^{-1} \mid -X \mid S'\right]$$

i.e. if by some process we can change N into I, the same process applied to I will give the required inverse N^{-1} and applied to C will give the solution (with signs reversed) of the equations.

The process operates on the combined matrix $\left[N \mid I \mid C \mid S\right]$ as follows

(a) reduce a diagonal element of N to 1 by dividing the whole row in which it occurs by its original value

(b) reducing the other elements of N in the same column as this diagonal element to zero by subtracting from all the elements of a row, an appropriate multiple of the corresponding elements of the row dealt with in (a)

(c) repeating the process of (a) and (b) until the unit matrix is obtained instead of N

The procedure is best illustrated by a numerical example and for this the Normal Equations dealt with in Section 9 will be used

$$\left[\begin{array}{ccc|ccc|c|c} +.91410 & -.53574 & +1.55168 & 1 & 0 & 0 & -30.76455 & -27.83451 \\ -.53574 & +.69168 & -.30966 & 0 & 1 & 0 & +9.03667 & +9.88294 \\ +1.55168 & -.30966 & +5.00000 & 0 & 0 & 1 & -68.50000 & -61.25798 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc|c|c} +.43255 & -.43964 & 0 & 1 & 0 & -.31034 & -9.50626 & -8.82371 \\ -.43964 & +.67250 & 0 & 0 & 1 & +.06193 & +4.79447 & +6.08923 \\ +.31034 & -.06193 & 1 & 0 & 0 & +.20000 & -13.70000 & -12.25160 \end{array}\right] \begin{array}{l} R1-.31034\,R3 \\ R2+.06193\,R3 \\ R3 \div 5 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc|c|c} +.14514 & 0 & 0 & 1 & +.65374 & -.26985 & -6.37192 & -4.84294 \\ -.65374 & 1 & 0 & 0 & +1.48699 & +.09209 & +7.12932 & +9.05462 \\ +.26985 & 0 & 1 & 0 & +.09209 & +.20570 & -13.25848 & -11.69084 \end{array}\right] \begin{array}{l} R1+.65374\,R2 \\ R2 \div .67250 \\ R3+.09209\,R2 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc|c|c} 1 & 0 & 0 & +6.88990 & +4.50420 & -1.85924 & -43.90189 & -33.36737 \\ 0 & 1 & 0 & +4.50420 & +4.43157 & -1.12337 & -21.57108 & -12.75895 \\ 0 & 0 & 1 & -1.85924 & -1.12337 & +.70742 & -1.41155 & -2.68665 \end{array}\right] \begin{array}{l} R1 \div .14514 \\ R2+4.50420\,R1 \\ R3-1.85924\,R1 \end{array}$$

It should be noted that the elements of the checking column in the original matrix are 1 more than those in the example of Section 9. This is due to the fact that the unit matrix I has been included in the composite matrix as well as N and C. At the side of each row of the subsequent matrices is a statement showing how the figures were obtained, the letters R 1, R 2, and R 3 referring to the elements of the first, second and third rows respectively of the matrix immediately above that for which the calculations are being made.

Here again the calculations should be made a row at a time and each row should be checked by the checking term as compared with the

sum of the remaining elements of the row, any significant discrepancy being investigated. It should also be noted that the symmetry of the original Normal Equation coefficient matrix is partly maintained at each stage of the procedure if we consider a matrix formed by superimposing the equivalent of the I section over that of the N section, ignoring in both cases the columns involving only O's and 1. For the third matrix this would result in

$$\begin{bmatrix} +.14514 & 0 & 0 \\ -.65374 & 1 & 0 \\ +.26985 & 0 & 1 \end{bmatrix} \& \begin{bmatrix} 1 + & .65374 & -.26985 \\ 0 + & 1.48699 & +.09209 \\ 0 + & .09209 & +.20570 \end{bmatrix} \text{ combining to give}$$

$$\begin{bmatrix} +.14514 & +\ .65374 & -.26985 \\ -.65374 & +1.48699 & +.09209 \\ +.26985 & +\ .09209 & +.20570 \end{bmatrix}$$

and in this last matrix it will be seen that $a_{ij} = a_{ji}$ if both elements come from the same matrix and that $a_{ij} = -a_{ji}$ if they come from different matrices. These facts can be used either as additional checks or alternatively to save a certain amount of computation.

The final matrix gives the inverse matrix

$$\begin{bmatrix} +\ 6.88990 & +\ 4.50420 & -\ 1.85924 \\ +\ 4.50420 & +\ 4.43157 & -\ 1.12337 \\ -\ 1.85924 & -\ 1.12337 & +\ \ .70742 \end{bmatrix}$$

and also the solution

$$\begin{bmatrix} +43.90189 \\ +21.57108 \\ +\ 1.41155 \end{bmatrix}$$

Some authorities recommend dealing at each stage with the largest element of what remains of the N matrix, interchanging the rows or columns if necessary in order to bring this element on to the main diagonal, the idea being to avoid the rapid accumulation of errors which might occur if a row had to be divided by a small num ber. In survey adjustments this interchanging of rows is not necessary as the diagonal terms are, except in rare cases, dominant.

B5. Adaptation of the Cholesky Method to give the Inverse Matrix

This is done by inverting B, the Cholesky upper triangular matrix and then finding the inverse of the normal equation matrix from

$$N^{-1} = B^{-1}(B^T)^{-1} \qquad \text{B5.1}$$

B^{-1} will be an upper triangular matrix just as B is and if its elements are t_{ii} on the main diagonal and t_{ij} above it, then from $BB^{-1} = B^{-1}B = I$ the following relationships are obtained

$$b_{ii}\, t_{ii} = 1 \quad \text{and} \quad \left[b_{ik}t_{kj}\right]_{k=i}^{j} = 0 \qquad \text{B5.2(a) \& (b)}$$

and the values of t_{ii} and t_{ij} can be calculated from these formulae. Subsequently if m_{ij} are the elements of the inverse matrix N^{-1} (a symmetrical matrix of order n), then equation B5.1 gives the formulae

$$m_{ii} = t_{ik}^{2}\Big|_{k=i}^{n} \quad \text{and} \quad m_{ji} = m_{ij} = \left[t_{ik}t_{jk}\right]_{k=j}^{n} \quad j > i \qquad \text{B5.3(a) \& (b)}$$

The example of Section 5.2 will be re-worked in order to illustrate the calculation processes.

The mathematical operations $B^{-1}B = I$ and $B^TB = N$ are similar in general procedure and in practice, when the inverse matrix is required it is usual to carry both processes out simultaneously, the top right hand portion of the tabulation below giving the calculations for B from N and the bottom left hand portion giving those for $(B^{-1})^T$ from I. A checking term for this second portion is included on the left hand side.

I and $(B^{-1})^T$		N and B				
Σ					K	S'
1	1	3.05	-1.5	- .75	+.0900	+.8900
.5726	.5726	1.7464	- .8589	- .4295	+.0515	+.5096
1	0	1	3.25	-.75	+.0825	+1.0825
.9412	.3103	.6309	1.5850	-.7059	+.0800	+ .9591
1	0	0	1	2.70	-.1725	+1.0275
1.3451	.3274	.3136	.7041	1.4203	-.0661	+1.3542

The method of calculation of the left hand portion of this

tabulation is shown below for the elements of the last row

$$.7041 = 1/1.4203$$

$$.3136 = (0 - .6309 \times (-.7059))/1.4203$$

$$.3274 = (0 - .5726(-.4295) - .3103(-.7059))/1.4203$$

$$1.3451 = (1 - .5726(-.4295) - .9412(-.7059))/1.4203$$

Having obtained the elements of B^{-1} by this method, the required inverse matrix N^{-1} is obtained by use of equations B5.3(a) and (b) e.g.

$$m_{22} = .6309^2 + .3136^2 = .4963$$

$$m_{12} = .3103 \times .6309 + .3274 \times .3136 = .2984$$

and the complete inverse matrix will be found to be

$$\begin{bmatrix} .5313 & .2984 & .2304 \\ .2984 & .4963 & .2207 \\ .2304 & .2207 & .4957 \end{bmatrix}$$

B.6 Formation of the Inverse Matrix in the Gauss-Doolittle Method

In this method N is put equal to B^TDB and hence

$$N^{-1} = B^{-1}D^{-1}(B^T)^{-1} \qquad \text{B6.1}$$

where B is the Gauss-Doolittle coefficient matrix and D is a diagonal matrix, the elements of which are the inverses of the diagonal elements of B and the diagonal elements of D^{-1} will thus be b_{ii}

The inverse B^{-1} will be obtained from the Gauss-Doolittle matrix B by exactly the same process as in Section B.5, that is by use of the equation B5.2(a) & (b). The normal equation inverse will however be obtained in a different way since equation B6.1 is now being used and the appropriate relationships in this case are

$$m_{ii} = \left[b_{kk}t_{ik}^2\right]_{k=i}^{n} \text{ and } m_{ji} = m_{ij} = \left[b_{kk}t_{ik}t_{jk}\right]_{k=j}^{n} \quad j > i \qquad \text{B6.2(a) \& (b)}$$

Starting with the same set of Normal Equations as were used in the last section, the calculation of the upper triangular matrix and its inverse, by this method will be

I and $(B^{-1})^T$ $\sum$		B			K	S'
1	1	3.05	-1.5	- .75	+.0900	+ .8900
.3279	.3279	-1	+ .4918	+ .2459	-.0295	- .2918
1	0	1	2.5123	-1.1189	+.1268	+1.5202
.5938	.1958	.3980	-1	+ .4454	-.0505	- .6051
1	0	0	1	2.0172	-.0939	1.9234
.9470	.2305	.2208	.4957	-1	+.0465	- .9535

the calculation processes for the elements of the last row of $(B^{-1})^T$ and the checking term being

$$.4957 = 1/2.0172$$

$$.2208 = (0 - .3980(-1.1189))/2.0172$$

$$.2305 = (0 - .3279(-.75)-.1958(-1.1189))/2.0172$$

$$.9470 = (1 - .3279(-.75) - .5938(-1.1189))/2.0172$$

The inverse of the normal equation matrix is now found by equations B6.2(a) & (b) e.g.

$$m_{22} = .3980^2 \times 2.5123 + .2208^2 \times 2.0172 = .4963$$

$$m_{12} = .1958 \times .3980 \times 2.5123 + .2305 \times .2208 \times 2.0172 = .2984$$

and the complete inverse matrix will be found to be the same as before.

APPENDIX C

LEAST SQUARE ADJUSTMENT IN OTHER SURVEY DISCIPLINES

Least square adjustment can be applied in any situation in which more observations have been made than are necessary to determine the required unknowns and in photogrammetry its main use is in the derivation of ground co-ordinates from machine co-ordinates by a similarity transformation, but it can also be used in minor computations such as parallax bar heighting.

Both these operations come under the general heading of curve fitting, in which it is desired to represent a series of observations by a particular mathematical formula, the unknowns being the constants in the formula. As an example consider parallax bar heighting using the formula

$$\text{Ground Height} - \text{Parallax Bar Height} = a_0 + a_1\,x + a_2 y + a_3 xy + a_4 x^2$$

where x & y are the photo co-ordinates and information is available for 7 control points.

Control Point	Photo Co-ordinates		Parallax Bar Height	Ground Height
	x_{mm}	y_{mm}		
A	10	85	237.1	237.1
B	71	83	198.6	180.7
C	42	7	174.7	200.7
D	65	-80	250.0	271.8
E	12	-77	246.4	230.0
F	32	41	240.0	265.0
G	50	-17	270.0	292.3

It is obvious that, using these figures, the coefficients of a_3 and a_4 in the observation equation will be very much larger than those of a_0 and this effect will be squared in the normal equations resulting in a very unbalanced normal equation matrix. For this reason it is desirable to use different units in order to make all coefficients of approximately the same order and in this case we will work in decimetre units. The observation equations will then be -

a_0	a_1	a_2	a_3	a_4	Constant
1	.10	.85	.0850	.0100	0
1	.71	.83	.5893	.5041	+ 17.9
1	.42	.07	.0294	.1764	- 26.0
1	.65	-.80	-.5200	.4225	- 21.8
1	.12	-.77	-.0924	.0144	+ 16.4
1	.32	.41	.1312	.1024	- 25.0
1	.50	-.17	-.0850	.2500	- 22.3

giving the normal equations

a_0	a_1	a_2	a_3	a_4	C
7.0000	2.8200	.4200	.1375	1.4798	-60.8000
2.8200	1.4798	.1375	.0896	.8671	-29.5630
.4200	.1375	2.8462	1.1188	.0896	+11.3900
.1375	.0896	1.1188	.6587	.0743	+18.2202
1.4798	.8671	.0896	.0743	.5370	-12.6723

The solution of these equations is

$$a_0 = -37.2821 \qquad a_1 = 297.0451 \qquad a_2 = 19.1957 \qquad a_3 = -53.5520$$
$$a_4 = -349.0953$$

and these constants are applicable if the photo co-ordinates are measured in decimetres. For working in millimetres, the a_1 and a_2 values should be multiplied by 10^{-2} and the a_3 and a_4 values should be multiplied by 10^{-4}.

Least Square Adjustment is not often used in astronomical work but a simple example would be the determination of the clock correction at any moment in a timed series of astronomical observations. This is a further example of curve fitting, the curve in this case being a straight line and the formula to be used being

$$C_i = C_0 + R(T_i - T_0)$$

where C_i is the clock correction at time T_i , C_0 is the clock

correction at some datum time T_0 and R is the amount by which the clock correction changes per unit of time. R and C_0 are the unknown quantities which we have to find.

The observation equations will then be

$$v_i = C_0 + R(T_i - T_0) - C_i \qquad i = 1 \text{ to } n$$

and these give the normal equations

$$nC_0 + R[T_i - T_0]_1^n - [C_i]_1^n = 0$$

$$C_0[T_i - T_0]_1^n + R[(T_i - T_0)^2]_1^n - [C_i (T_i - T_0)]_1^n = 0$$

For hand calculations the arithmetic can be simplified by choosing T_0 as the mean value of the observed T_i and if this is done then $[T_i - T_0]_1^n = 0$ and the normal equations reduce to

$$n C_0 - [C_i]_1^n = 0$$

$$R [(T_i - T_0)^2]_1^n - [C_i (T_i - T_0)]_1^n = 0$$

giving C_0 and R directly.

As a numerical example consider the following set of observations

Eastern Std Time			Observed Clock Time			Clock Correction
h	m	s	h	m	s	secs.
17	50	28.0	17	49	45.0	+ 43.0
18	59	31.0	18	58	59.7	+ 31.3
19	14	08.5	19	13	39.7	+ 28.8
19	16	08.5	19	15	40.1	+ 28.4
19	47	31.6	19	47	08.3	+ 23.3
19	56	30.4	19	56	08.6	+ 21.8

The calculation is carried out as follows

T_i	c_i	$T_i - T_0$	$c_i(T_i - T_0)$	$(T_i - T_0)^2$
17.829	+ 43.0	- 1.339	- 57.5770	1.7929
18.967	+ 31.3	- .201	- 6.2913	.0404
19.228	+ 28.8	+ .060	1.7280	.0036
19.261	+ 28.4	+ .093	2.6412	.0086
19.786	+ 23.3	+ .618	14.3994	.3819
19.936	+ 21.8	+ .768	16.7424	.5898
115.007	176.6		- 28.3573	2.8172

$T_0 = 115.007 \div 6 = 19.168$

$C_0 = 176.6 \div 6 = 29.43$ seconds

$R = -28.3573 \div 2.8172 = -10.066$ seconds per clock hour

N.B. The simplification of the arithmetic by transferring the origin to the centres of gravity of the systems of known values can be used in all cases of curve fitting but its value is most apparent when the curve is linear as in the above example or in similarity and affine transformations.

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