LINEAR MEASUREMENTS
(Note for Students)

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INTRODUCTION - Unit of Measure

The aim of this note is to fill a gap in textbooks, which while dealing in some detail with "chain surveying", generally an obsolete method nowadays, do not give enough information about the measurement of short distances especially in engineering surveys.

The physical quantities measured in surveying are two, namely angles and distances. The latter consist of linear measurements along a straight line between two points. There are direct and indirect methods of linear measurement in surveying. The distance is determined in terms of the known length of some material or the known wavelength of some electromagnetic vibration etc. The angles are measured by means of an instrument so that the sine rule (see figure 1)

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{b}{\sin\gamma}$$

can be solved. In these notes only the direct measurement by comparison with a material standard is dealt with in some detail.
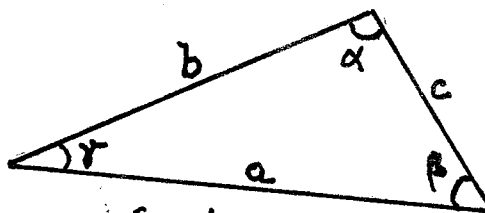

Fig. 1

The usual range of linear measurements in engineering and general surveying consists of distances from a few metres to kilometres at most. The unit of measure will be taken as the metre. It is usual in practice to omit the abbreviation m for every measurement especially in technical documents such as field books where it would be totally unnecessary. The metre is very useful in that it is easy to jump from this unit to any of its multiples or submultiples. When these are used, s for instance in estimating errors or applying corrections (see page 7 et seq.), the new unit, say centimetre, must be clearly indicated.

It might be necessary to have to deal, in some cases, with measurements made in imperial feet, which are equal to metres multiplied by 0.3048. This, however, should not be difficult to do because the foot, recently anyhow, has been divided decimally. If the measurement, however, is made in links, rods, chains as occurs in some obsolete survey measuring instruments, it is probably easier to transfer all the data into metric values before attempting any manipulation.

BRIEF DISCUSSION OF ERRORS IN LINEAR MEASUREMENTS

Linear measurements are affected by both systematic and accidental error. Direct measurement is made by means of usually steel band or tape which has been standardised and its corrections determined (see page 5 and following pages). The corrections are to be applied whenever the tape has been used for measurement, except when only a gross approximation is required.

Accidental or random errors are produced by the sum of imponderable causes, the main one, in our case, being the material difficulty of holding the standard in alignment steadily with the distance to be measured. On the other hand, one linear measurement is one observation, and provided that no systematic part is included the most straightforward application of the theory of errors is fully valid in this case. Such theory is not the object of these notes, but by taking a sufficiently large number of measurements of the same length, naturally with the same standard and under similar conditions, the arithmetic mean of the measurements is the most probable value of the length and the relevant accuracy may be evaluated by obtaining the standard deviation or standard or probable error, whichever is more convenient.

As a matter of fact such procedure is seldom applied in general surveying because it requires a large number of observations to be valid, and a large number of observations is usually not possible on account of the economics of the task. The common practice is to take two or a few more measurements as an insurance against gross error or mistakes rather than a way to improve accuracy.

In fact the methods and field procedures used are such that the standard deviation of a single observation suits the accuracies required within a given confidence limit for the purpose of the particular task. The surveyor however, as well as the engineer must be prepared to deal with special cases according to their merits.

The above statement that each linear measurement makes one observation needs some criticism. The corrections aimed to eliminate systematic errors require several auxiliary observations such as slope, temperature, tension to be made. To each of such observations, the errors theory should be applied and this fact should not be overlooked as a principle. However, with the possible exception of slope, the corrections are of several orders of magnitudes smaller than the whole distance to which they apply. Therefore in most cases the standard deviation acceptable for the measurements made, is quite large even in the case of very accurate distance measurement. When the error theory is not fully applied, or say when only one or two observations are made, the use of more than the required number of decimals should be avoided. If when the standard requires centimetres to be read to give the required accuracy there is no use in recording millimetres produced by the averaging process. These unnecessary decimals are now all too easily provided by the pocket computers being used. Theoretically such figures are not obnoxious, but they are the possible cause of mistakes later and are of no practical use.

COMPARISON BETWEEN DIRECT AND INDIRECT MEASUREMENTS

Direct methods are simple to apply, even if they are sometimes difficult to carry out. They do not require the measurements of quantities other than the distance itself with exception of small correction values. Computations are simple and the chance of gross errors remote. Also, direct measurement allows a far greater accuracy than obtained from the measurement of indirect distances. On the other hand, direct measurements may be applied only over a limited range. Both ends of the line to be measured should be accessible and the whole line should be free of obstructions. Indirect measurements can overcome all these limitations and this is the reason for the great use of such methods especially in the recent past. Nowadays, the introduction of electronic distance measurement devices has enormously extended the range of applicability of direct measurements and partially eliminated the need of indirect methods.

REVIEW OF INDIRECT METHODS

The most powerful and flexible method of indirect distance measurement is that of triangulation. This is an important part of the surveying science in itself and it is well known as a method of finding the coordinates of points rather than a method for measuring the distance between them, but the two things are the same in concept.

Triangulation consists of the application of the sine rule as outlined on page 1, where the triangle starts with the length for one side known and the measuring of at least two of its angles, this allows the two remaining sides to be determined and to be used for sides for the starting of a new triangle and so on without limit. The concept is extremely simple and so are the relevant computations, at first. The complications arise from the fact that triangulation is carried out usually on the earth's surface, which is not a plane and therefore the sides of large triangles are not quite straight lines.

This technique is still largely applied in topographical surveying, less often in engineering surveying, and in this case only over small triangles; as a result complications are avoided. More common in short range survey are a number of methods based on the same principle as triangulation but simplified in order to use specific triangles, (rectangular, isosceles) with one side made up of some part of the instrument. Among these methods, the commonest is tacheometry which consists in observing the intercept of a graduated staff subtended by a fixed angle. This method will be dealt with in another part of the course. Sometimes, especially in modern engineering surveying instruments, the subtense angle in tacheometry is not fixed but is manipulated by some mechanical devices in order to read the horizontal distance directly instead of the subtense length.

Almost obsolete, but still, a method of theoretical interest is the subtense bar. It consists of two arms fixed at the end of an horizontal bar, whose known length is known with high accuracy. The centre of the bar is plumbed over one end of the line to be measured and the bar held normal to such line, while a very accurate angle measuring instrument is plumbed over the opposite end of the line. The angle subtended by the bar is measured accurately (see figure 2) and the distance is given by the formula

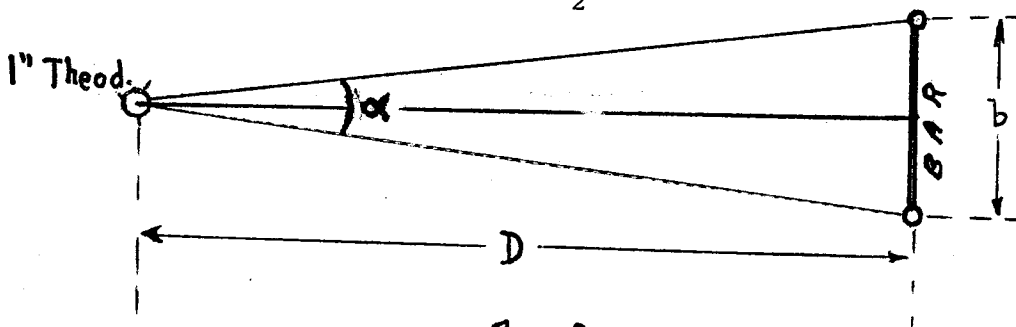$$D = \frac{b}{2 \ tg \ \frac{\alpha}{2}}$$

Fig. 2

Such a method offers an accuracy comparable to direct measurement over a limited length and has the advantage of being quicker and simpler than measurement with a material standard. However, it requires the use of an expensive precision angle measuring instrument, which is of not much use in the other operations of a short range survey.

When only a limited accuracy is required, e.g. military applications, the subtense system is applied somewhat in reverse and the bar and the angle measurer are combined in one instrument (the Rangefinder). In this the two arms of the bar are substituted by two telescopic objectives, one of them can be rotated to bring the image of the object formed into superimposition with the corresponding image formed by the other (fixed) objective. The rotation is usually measured directly as units of the distance from the Rangefinder to the object.

For long distances such as required in the Navy, up to tens of kilometres, large base rangefinders with fixed objectives are used and the coincidence of the images formed is done by the eyes of the operator, to whom the images are brought separately as in stereoscopy. The relevant details are unimportant because this method even though interesting is now replaced by RADAR.

DIRECT DISTANCE MEASUREMENT: REVIEW OF E.D.M.

Three different types of device are used for direct distance measurement: 1) comparison with a material standard, 2) kinematic devices and, 3) electronic devices.

The first system, even if it is the most important in a basic course of surveying, requires little explanation. The material standard can be of a rigid nature (rods, Bessel's bars), but these have now been abandoned in surveying, because of limited lengths of each of the standard and the cumbersome method of measuring in short legs. Standards of flexible nature are, in order of sophistication, chains, cloth tapes, steel tapes, steel bands and Invar wires.

The surveying chain is described by most of the surveying books still in circulation, therefore does not deserve any further mention except that it is no longer in existence. Cloth and plastic tapes are well known and largely used in various trades and also if they are of a reliable type, in surveying. They are used for preliminary measurements or very effictively over very short ranges of a few metres. They cannot be used for longer distances, however, because their behaviour under the stress needed to stretch them is unpredictable.

For distances up to a few tens of metres, usually no more than 30, the most convenient device is the steel tape, made of a high Nickel steel, stainless and with a small coefficient of thermal expansion (see page 6). It is sufficiently strong and light to have negligible corrections for tension and sag.

Steel bands of 100 m (less frequently 200 m) in nominal length are largely used in engineering surveying. They have a rectangular section with a sectional area of about 1-2 mm^2. They are made of ordinary steel and wound over suitable drums. At one end they have a reader, which is nothing more than a steel band, usually 2 or 3 m long, graduated in millimetre while the whole length is graduated every one or two metre. These devices, if not used carefully, are much less reliable than good steel tapes owing to their sensitivity to thermal and elastic effects. However, the behaviour of a piece of ordinary steel that is armed with about 0.5% of Carbon, with a cross section of insignificant size relative to its length, behaves under heat and tension in a manner well known and accurate correction can be made for these effects as will be seen in the following pages.

For more precise measurement, and sometimes as an alternative to steel bands, Invar wires are in use. Invar is an alloy (64% Fe and 36% Ni). It has more or less the same mechanical characteristics as that of steel but a coefficient of thermal expansion of about one eighth that of steel. This under ordinary conditions has often been neglected entirely.

Since the main advantages of the invar wires (usually of circular section with cross sectional area of about 1 mm^2) is to eliminate corrections, they are supplied in short lengths, 24 or 30 m, in order to keep down to a negligible degree corrections for tension and sag. However when high precision is required, invar wires are standardised and corrected in much the same way as steel bands are.

Kinematic devices for distance measurements are very common but seldom used in surveying, because of their gross approximation. A common one is the Odometer or Speedometer attached to most vehicles, this is sometimes useful for preliminary measurements in the field. In the past a few types of precise Odometers have been used with some success. As a curiousity it can be mentioned that until recently one could buy a PODOMOTER (which counted paces) to be fixed to the leg of the surveyor or preferably of a pack animal, which is not subject to psyciological interferences. This is not altogether a laughing matter, since three hundred years before Christ the Arc of Meridian was measured by Eratostenes with astonishing accuracy done by counting the paces of a camel through the Egyptian desert.

Of increasing importance are the electronic devices for linear measurements, nowadays they are used almost exclusively for distances over a few hundred metre. The standard is given by an appropriate electromagnetic vibration which travels between two stations (Tellurometer) or between a station and a reflector (Geodimeter) placed at the ends of the distance to be measured. The linear distance is obtained from the value of the time taken for the wave to travel between the two end points of the line and back again. Generally, this is measured by method of phase comparison. The modulator vibration can be MF radio waves of length 10 m or less, down to infrared and visible light of incoherent or coherent (LASER) nature.

STANDARDISATION OF A STEEL BAND

It has been said that a steel band has a Nominal length. This is the true length of the new band, at nominal temperature and tension. These are usually marked at one end of the band.

In time and with the use of the band, its length often changes and also because the band is rather expensive, if it is accidentally broken it can be conveniently repaired but the joint always produces a slight alteration of length. Therefore at the beginning, and possibly at the end of any major job, the steel band should need standardisation.

For the standardisation of a band a length of the same order of magnitude as the nominal length of the band is required. This length should be known with the greater accuracy than that required by the eventual measurements.

It is known that any object, in our particular case, a steel band, elongates with any variation of temperature Δt, as given by the formula:

$$\Delta l_t = \alpha.l.\Delta t \tag{1}$$

where α, called the coefficient of thermal expansion, has a value, for ordinary steel of 1.1 to 1.2 x 10^{-5}, per C^o (Celsius degree).

Similarly, for any variation of tension or pull F, the length of the band varies in proportion to the value given by:

$$\Delta l_F = \frac{l\Delta F}{a.E} \tag{2}$$

where E is called the <u>Modulus of Elasticity</u> or <u>Young's Modulus</u> and has the value, for ordinary steel of about 2 x 10^5 N/mm2; and a is the sectional area of the band.

Provided that the band is stretched along the base (which may be another band, already standardised) and provided that the band is subject only to a straight pull and not for instance to any pull from its own weight, there are no other factors influencing its actual length.

Now let us suppose the base of length L, is getting measured with our steel band and the <u>crude</u> length of this band is 1, while it is under a tension F. At the same time the temperature of the environment T is measured, and it is assumed that the band on account of its small thermal capacity, is at the same temperature. This leads to the equation:

$$L = 1 + \Delta l_t + \Delta l_F \tag{3}$$

But up to now we have not taken into account any temperature or tension variation. In general the tape's nominal length is not correct. It is not possible to regraduate the tape but we may assume a suitable fixed value of the tension; and then determine the temperature at which the incorrect band assumes its correct length. The temperature at which this occurs may be very different from the temperature at which the tape is used but this standardisation or dummy temperature effects only the calculation and not the metal of the tape. Standardisation then consists in finding such a dummy temperature. In equation (3) since the tension is kept constant $\Delta l_F = 0$ and if we consider $1 - L$ as positive we obtain

$$\Delta t = \frac{\Delta l_t}{1.\alpha} \tag{3'}$$

This quantity Δt is added to the ambient temperature T at the moment of calibration, to give the standisation temperature $T_c = T + \Delta t$, which is the temperature at which the band would measure correctly, provided the tension is that fixed during the standardisation. At this time then, the surveyor notes in his field book under the date of the day the important heading "Calibrated length of the Band: 100.000 m at (T_c) C^o and (F) N.

PROCEDURE FOR MAKING LINEAR MEASUREMENTS WITH STEEL BANDS:
TEMPERATURE AND TENSION CORRECTIONS

The steel band is stretched between the two ends of the distance to be measured. At one end the fiducial mark representing the nearest metre (or two-metre mark) on the band is brought into coincidence with the end of the line (centre of a peg or whatever it is). At the other end of the line the second operator reads on the graduated reader taped the fraction of metre,

usually to the nearest millimetre. The whole operation is repeated if necessary (see page 2), and the sum of the reading of operator A (i.e. the whole metre of two-metre reading) plus the reading of operator B (the fine reading) is taken as the crude distance "1".

At the same time the ambient temperature is recorded and also the tension. For the temperature an ordinary theromometer is used, and the temperature reading is taken to the nearest degree. This is quite sufficient even for high precision work. The temperature is easily taken, but the operator must use his intelligence. The band assumes the ambient temperature fairly quickly but, if the band is partly exposed to the hot sun, and partly to cool shadow, precautions must be taken.

The tension of the tape is often measured by means of a spring balance, held by either of the operators at one end or in some rare instances, by each of the two operators at each end. From the third law of Dynamics both spring balances should read the same tension.

For obtaining the true distance, equation (3) is applied, and terms Δl_t and Δl_p are obtained from equations (1) and (2) respectively where Δt is the difference (with its sign) between the measured temperature and the Standardisation Temperature, and P is the difference (with its sign) between the pull of the measurement and the pull applied during standardisation.

It should be noted that both temperature and tension corrections may be either positive or negative, they are positive when the temperature (tension) of the measurements are higher than those of standardisation, and negative vice versa.

SAG CORRECTION

So far we have taken it that the band is stretched over distance and fully supported along its length so that its own weight has no effect. This is not the usual case in practice, more often the band is wholly or partly suspended over the distance, usually because one is measuring over an irregular ground surface (see figure 3).

Fig.3
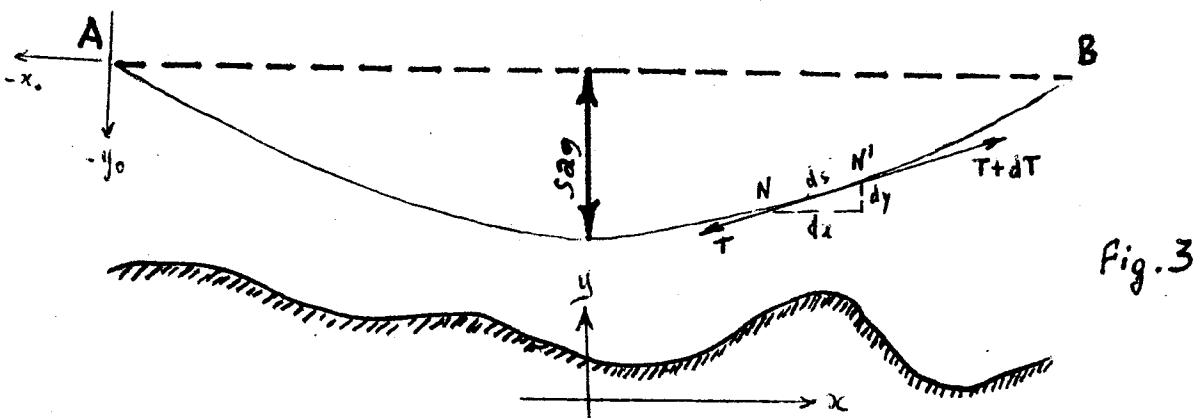
It is readily seen that, however much the band is pulled the suspended part never becomes a straight line but forms a curve in its vertical plane and that curve is called a Catenary. It is considered that a flexible wire such as a survey band, when freely suspended, approximates very closely to a PARABOLA. However this geometrical approximation does not offer an easy way of finding a distance between the two suspension points, or, i.e. the chord of the Catenary. Therefore, some dynamic considerations must be considered.

The static model sketched in figure 3, shows two points N and N', at an infinitesimal distance ds apart, on these two points the forces T and T + dt act.

At the lowest point of the band (v) the acting force F is horizontal, and equal to the horizontal component of the forces T acting and any other point of the band. In fact the force T is equal to F plus the w eight of that part of the band included between V and the point considered. Such wieght for an infinitesimal length of the band is W.ds, where W is the weight of the unit length of the band and is constant for a homogeneous bnad of constant cross section. The total force acting at the point N of the system, in Vector notation is:

$$\underline{T} = \underline{F} + W\Sigma\underline{ds} \tag{4}$$

In the cartesian system where x is the horizontal axis, and y the vertical axis with V the origin we consider separately the horizontal and vertical components of T, viz. $T\frac{dx}{ds}$ and $T\frac{dy}{ds}$, from these two differential equations may be formed:

$$-\frac{dx}{ds} = F = cost \tag{5}$$

where F clearly is the scalar value of the Tension applied at each end of the band, in simpler words the force read on this spring balance holding the band, and

$$\frac{d}{ds}(T.\frac{dy}{ds}) = W\frac{dx}{ds} \tag{6}$$

which represent the variation of the component Weight along the band.

The integration of equations (5) and (6) gives T as a function of x and y, that is the Equation of the Catenary, in the classical form:

$$y = \frac{S}{2}(e^{x/S} + e^{-x/S}) = -S \cos h \frac{x}{S} \tag{7}$$

where S is the 'sag', see figure 3, and e is the base of the natural logarithms. (The symbol Cosh means "Hyperbolic cosinus")

Equation (7) enables us to solve x as a function of y in a catenary of known Sag. But the measurement of S with sufficient precision is totally impractical in our case, it is much easier to measure F, which is needed quite roughly, while the weight of the band is well known.

If F is much larger than the weight of the band, it is readily realised that the resulting catenary is rather 'flat'. Therefore the length of the band and the straight distance AB (figure 3) are not very different, but note however that it is exactly this difference which we are looking for. Therefore, for Infinitesimal length it may be assumed that the Differentials ds and ds of the arc and of the ascissa are equal:

$$ds = dx \quad and \quad \frac{dy}{ds} = \frac{dy}{dx} \tag{8}$$

and equation (6) becomes:

$$\frac{d}{dx}(T\frac{dy}{dx}) = W\frac{dx}{dx} , that is \quad T\frac{dy}{dx} = wx = cost \tag{9}$$

from which:

$$y = \frac{w}{T}\frac{x^2}{2} + cost \tag{10}$$

and by putting $T = F$ (the tension at the lowest point, or the constant pull along the band), equation (10) represents a parabola (see above).

By making the constant of integration in equation (10) equal to zero, the origin of the system is shifted to point A (figure 3) say $x = \phi$ for $y = \phi$. From the geometry of the parabola the

$$\text{(sag)} \quad S = \frac{W}{F} \cdot \frac{x^2}{8} \tag{11}$$

and, for the length of the parabolic arc AB

$$l = x + \frac{8S^2}{3x} = x \left(1 + \frac{w^2}{24F^2} x^2\right) \tag{12}$$

the expression $+\dfrac{w^2 x^2}{24F^2}$ x represents the Excess of length of the parabola with respect to its chord, or exactly the reverse of the Sag Correction. 'l' is a known value, the crude length of the band and therefore equation (12) can be solved for x. Rather than do this however we shall reconsider the assumption which led to equation (8) and because of the small difference between x and l, we can substitute x for l in the last term of equation (12), finally obtaining for the sag correction

$$C_s = -\frac{w^2 l^2}{24F^2} l = -\frac{w^2 l^3}{24F^2} \tag{13}$$

where the minus sign does not need explanation because it always applies.

The pull applied to a survey band to form an acceptable catenary is not sufficient to snap it (a rule of thumb says that about 2 newtons per metre). Long spans of unsupported catenaries of a tape should however be avoided because of the practical difficulties and also the large corrections produced. This can be done by supporting the tape between the end points.

By putting one or more supports along the band, two or more catenaries of shorter length are obtained. For simplicity the supports are usually placed equidistantly along the tape. Let us consider the simplest case of one central support. The sag correction of the whole system is given by twice the equation (13) applied over half the length, l:

$$C_{supp} = -2 \left(\frac{w^2 (l/2)^3}{24F^2}\right) = -2 \frac{\frac{w^2 l^3}{8}}{24F^2} = -\frac{\frac{w^2 l^3}{4}}{24F^2} = \frac{1}{4} C_s$$

In the case of two equidistant supports, the total correction is

$$C_{2\ supp} = -3 \frac{w^2 (l/3)^3}{24F^2} = \ldots = \frac{1}{9} C_s$$

From this the simple rule that the total sag correction of a length supported at equidistant intervals is equal to the sag correction of the whole length considered unsupported, divided by the square of the number of spans.

SLOPE CORRECTION

Usually all distances in surveying are referred to the horizontal plane. The distance measured by means of the tape is corrected for all relevant quantities. To obtain from this reduced distance, the horizontal distance, a further correction is needed, this required a knowledge of the slope angle, θ, of figure 4. In modern surveying this slope angle must not be confused with the vertical angle as measured by theodolites, as it will be seen in another part of the course that this value may not equal the angle θ.

The slope angle may be taken from the Field Book or from the more usually used instrument called a Clinometer. This is an instrument used for making slope corrections for distance measurements and for some other measurements occasionally. The angle is taken as the absolute value only, since it will be readily seen that the relevant correction is always negative.

Fig. 4
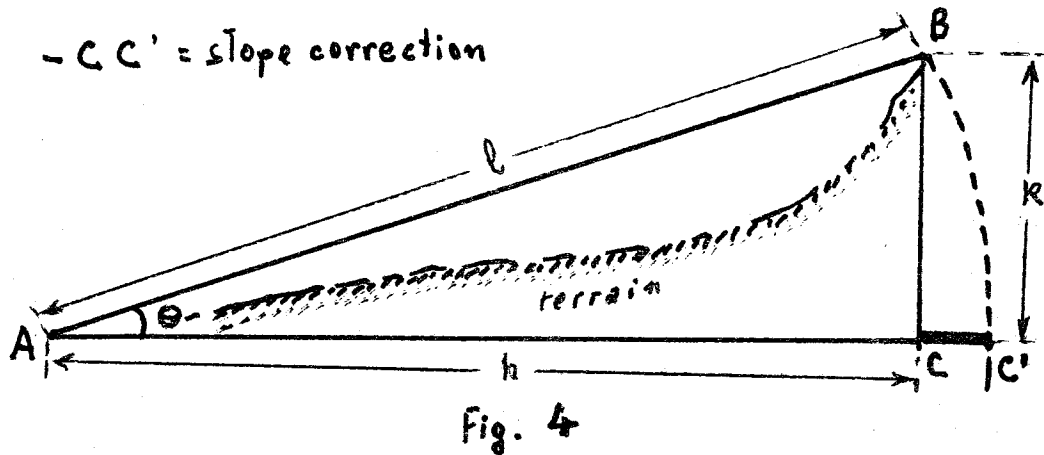
In figure 4, AB is the measured distance, reduced for its corrections and AC its horizontal projection which is the required value. It can be seen therefore that

$$H = l \cos \theta \qquad (14)$$

from which the value of the slope correction

$$l - h = l - l \cos \theta = l(1 - \cos \theta) \qquad (15)$$

Many books of Mathematical tables show the natural (or logarithmic) value of $(1 - \cos \theta)$ tabulated under the name of Versinus (versin). Its application is simple.

It should be noticed that if the slope angle is a large one the slope correction will not be small as compared with the true distance. In such cases, if high precision is required, the slope angle must be measured with greater accuracy or other sophisticated methods of reduction should be applied (e.g. Pythagoras' Theorem should be used after accurate levelling of points A and B).

ALIGNMENT CORRECTION

In some rare instances it might happen that the straight measurement AB is not possible, but the point B can be Off-Set in a new position C, a small distance from B and perpendicular to the line AB. This is the case of figure 4, seen in the horizontal instead of in the vertical plane. Obviously the alignment correction would be exactly the same as a slope correction. For a slope correction however it is easier to measure θ rather than BC, but for an alignment correction the reverse holds. Therefore it is easier to apply Pythagoras' theorem rather than the slope correction from angle θ. A simple and effective method for obtaining a value sufficiently close to the truth for all practical purposes of alignment correction, consists in expanding the cosine formula in a McLaurin series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \ldots \qquad (16)$$

(This can be used also for wroking out the slope correction.)
By substituting equation (16) up to its second term into equation (15) and
by equating arc BC - with the chord BC = k, the new relationship

$$1 - h = \frac{k^2}{2\,l} \tag{15'}$$

CORRECTIONS SEQUENCE

If all the corrections are very small, so that the same crude value
l may be used in all equations (1), (2), (13) and (15) without significant
alteration of the value of the various corrections, then the sequence is
unimportant and l may be rounded off to the nearest metre.

However this note illustrates the correct sequence for the various
corrections, and such a sequence should be applied when ever any of the
corrections is not small. The sequence is:

1. the band elongates with Temperature independently from any other action,
   that is, if it is hot the band is longer even on its reel: thus the
   temperature correction is applied first and the corrected length
2. is the original length of the band, which again elongates because
   of the pull, therefore the tension correction is applied second;
3. the parabolic length used to calculate the Sag correction, is the length
   of the band already heated and stretched, therefore this correction is
   applied third, and at this time the true length is obtained,
4. slope correction is applied to the true length obtained above, and
5. the alignment correction, if any, is applied last, provided that the
   off-set, as is usually done, is measured horizontally.